Fast Algorithms for Geometric Consensuses

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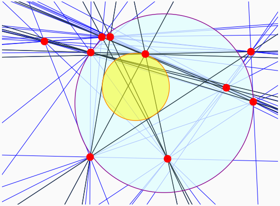
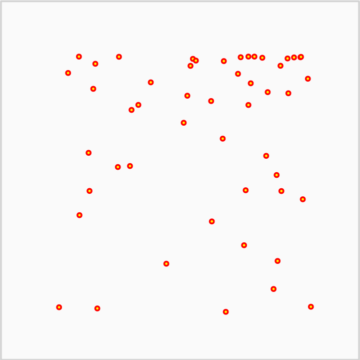
SoCG 2020, June 23–26

[1]University of Illinois at Urbana-Champaign

Student Presentation



The yolk: problem setup



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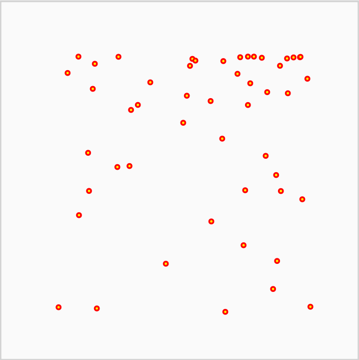

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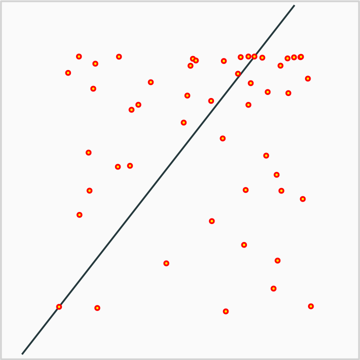

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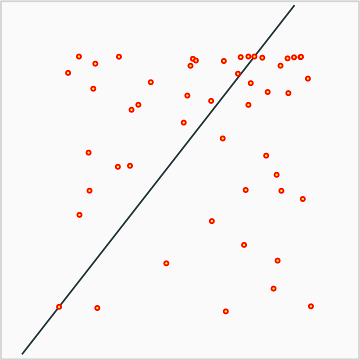

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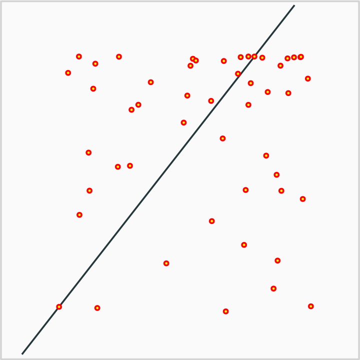

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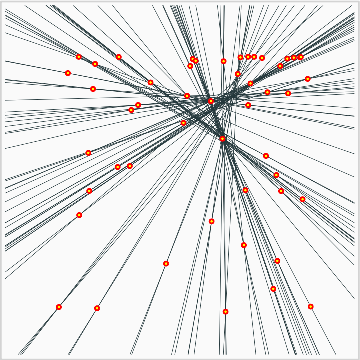

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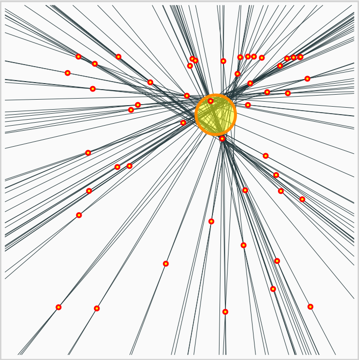

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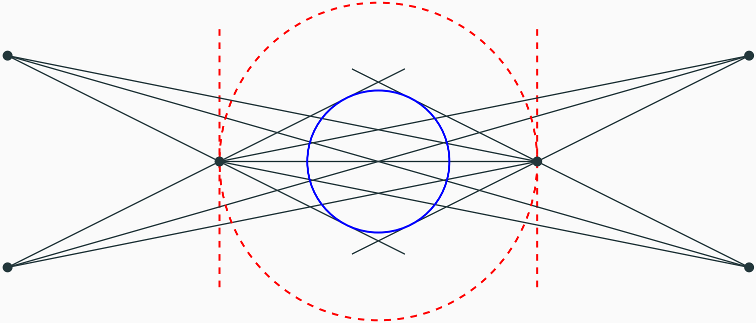
yolk $\neq$ extremal yolk [Stone and Tovey, 1992]



Previous work & our results

Exact:

- $O_d(n^{(d+1)^2})$ for yolk (improved to $O(n^4)$ in $\mathbb{R}^2$) [Tovey, 1992]

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In $\mathbb{R}^2$: $O(n \log n)$ expected time for yolk/extremal yolk, and $O_d(n^{d-1} \log n)$ for $\mathbb{R}^d$

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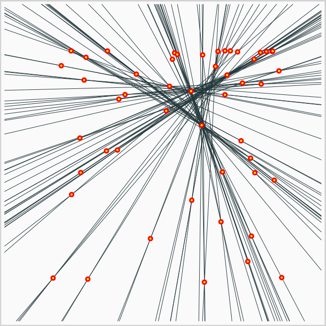

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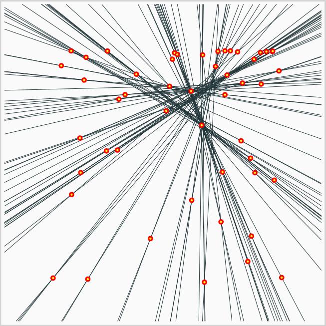

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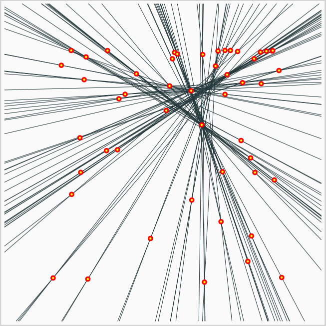

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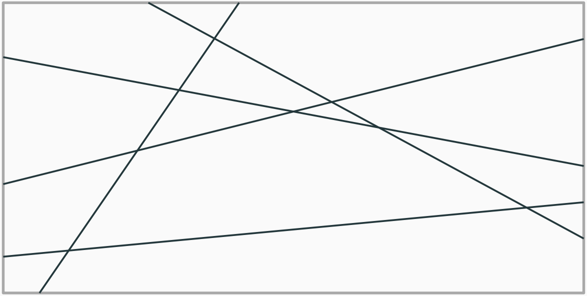
Definition: k-levels

Given lines L, k-level $=$ {points lying above or on k lines of L}.



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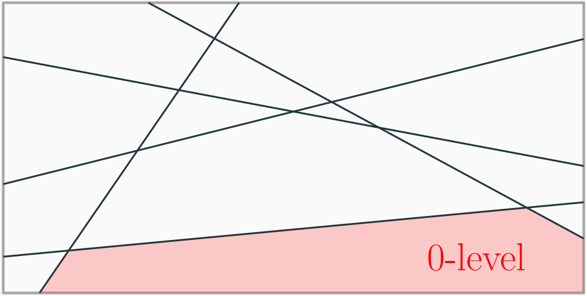

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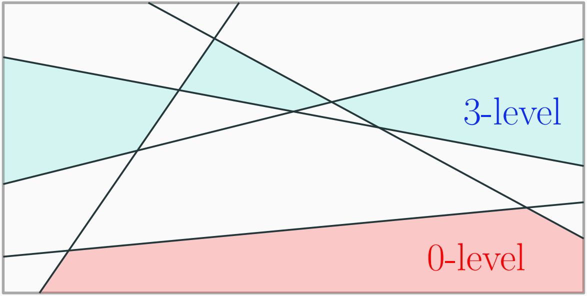

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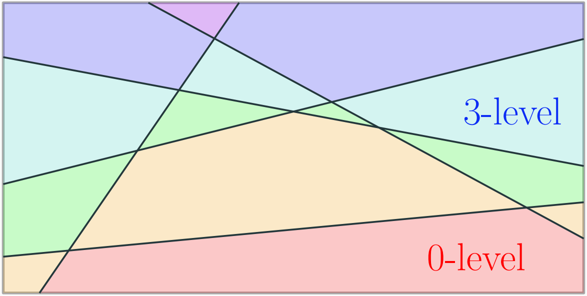

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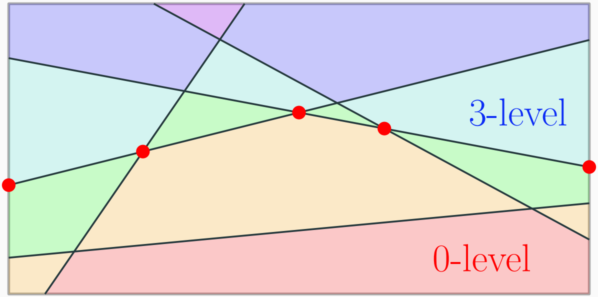
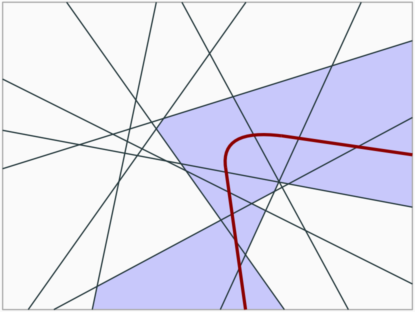


Definition: Zone of a surface

Given lines L, curve γ, the zone $\mathcal{Z}(\gamma, L)$ are cells of $\mathcal{A}(L)$ intersecting γ.



Lemma [Aronov et al., 1993, de Berg, Dobrindt, et al., 1995]

$\mathcal{Z}(\gamma, L)$ can be computed in $O(n \log n)$ expected time.

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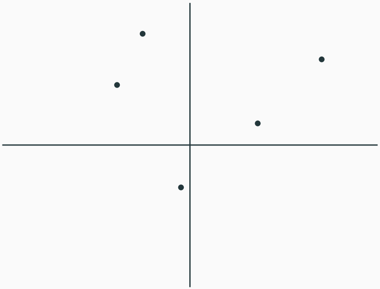

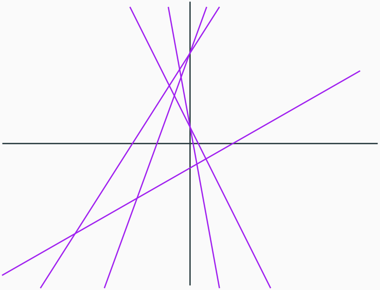

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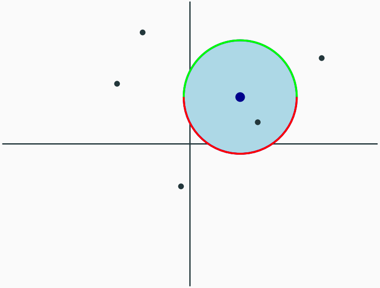

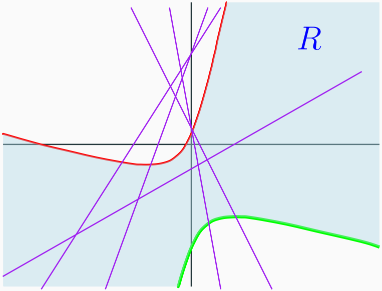

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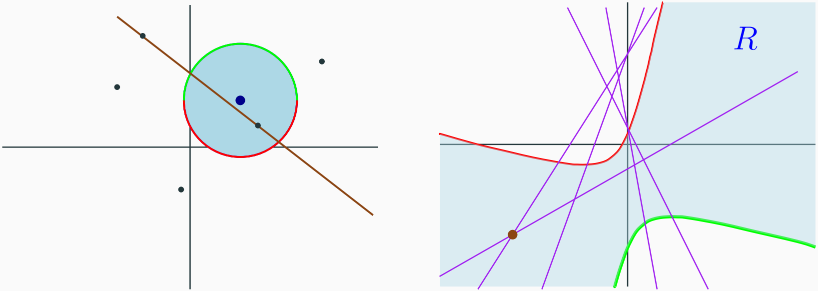
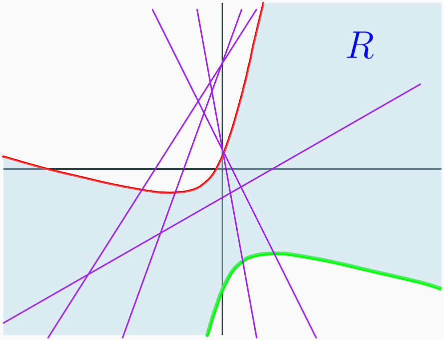
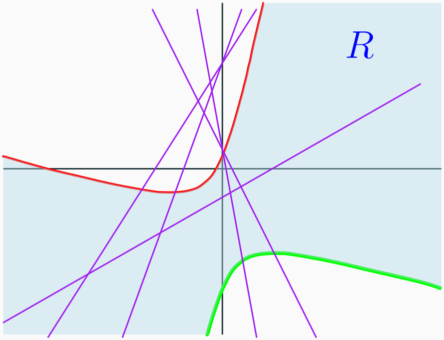
  - Extremal median line $\iff$ vertex of the $n/2$-level



The key subproblem



Is there a vertex of the $n/2$-level outside R?



Is there a vertex of the $n/2$-level outside R?

Check vertices of $\mathcal{A}(L(P))$ near boundary of R!

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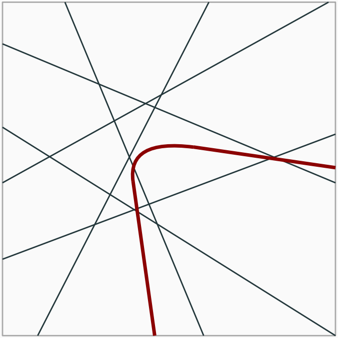
Task: Is there a vertex of $n/2$-level in $\mathbb{R}^2 \setminus R$?



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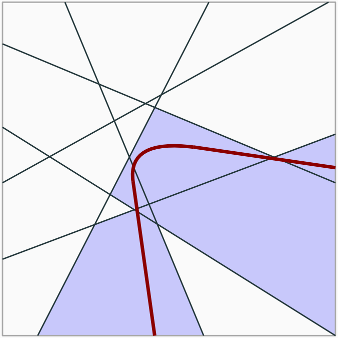

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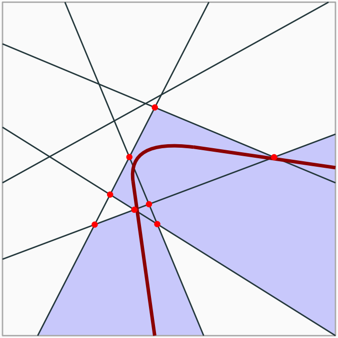

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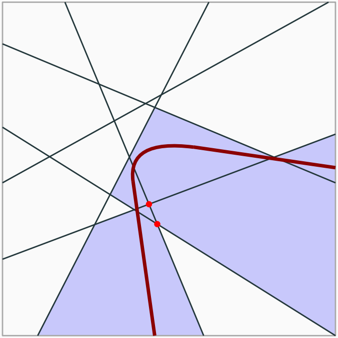

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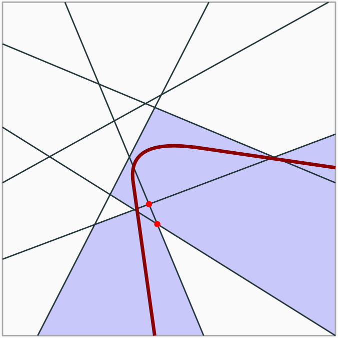

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Result: Decider takes $O(n \log n)$ time.

Solving the implicit LP

- P: a set of points; $n = |P|$
- $L(P)$: extremal median lines induced by P
- Goal: compute a solution for $L(P)$

For any $Q \subseteq P$, $m = |Q|$, assume in $D(m)$ time,

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Our result

$D(m) = O_d(m^{d-1} \log m) \implies$

extremal yolk in $O_d(n^{d-1} \log n)$ expected time.

Other applications I

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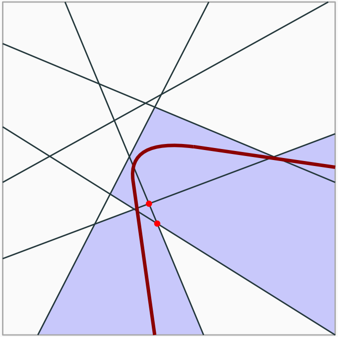

$\implies$ $O_d(n^{d-1} \log n)$ expected time

Other applications II

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- If $\mathcal{T}_k \neq \emptyset$, center ball = largest ball inside $\mathcal{T}_k$
- If $\mathcal{T}_k = \emptyset$, Tukey ball = smallest ball intersecting all halfspaces $H_k(P)$

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- E.g., $\mathcal{T}_1 = \text{conv}(P)$, $\mathcal{T}_{n/(d+1)}$ contains centerpoint of P
- If $\mathcal{T}_k \neq \varnothing$, center ball = largest ball inside $\mathcal{T}_k$
- If $\mathcal{T}_k = \varnothing$, Tukey ball = smallest ball intersecting all halfspaces $H_k(P)$

Our result

Center ball and Tukey ball can be computed in $O_d(n^{d-1} \log n)$ expected time

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Thank you!

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