# Active Learning a Convex Body in Low Dimensions

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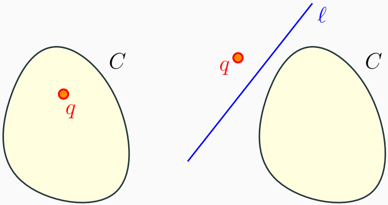
### Problem

**Input:** $P \subset \mathbb{R}^2$, oracle for unknown convex body $C$.

**Oracle:** Separation oracle.

**Goal:** Compute $P \cap C$ using fewest number of oracle queries.



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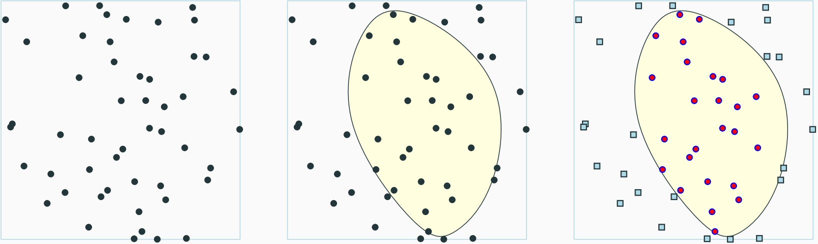

# Motivation: Active learning

- Input space X
- Learner data: $x_1, \ldots, x_n \in X$ (without labels)
- Learner can query oracle for label of any $q \in X$
- Build classifier using few queries
- What queries to choose?

# Additional motivation

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- Computational problems with oracle access:
    - Nearest-neighbor oracles [Har-Peled et al., 2016]
    - Proximity probe [Panahi et al., 2013]
    - Linear queries [Ezra and Sharir, 2019]

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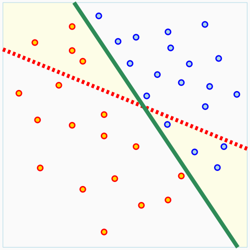

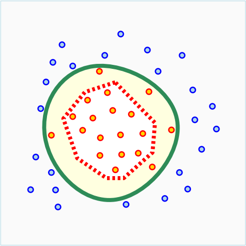

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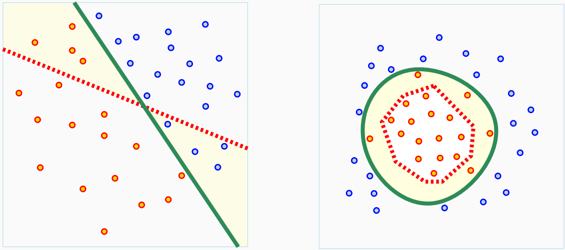

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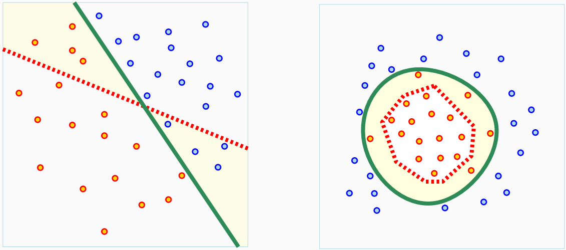

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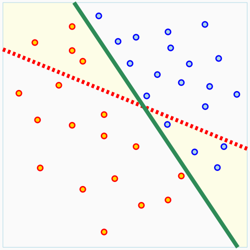
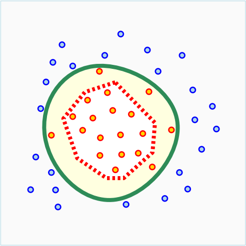
- Scheme fails for arbitrary convex regions





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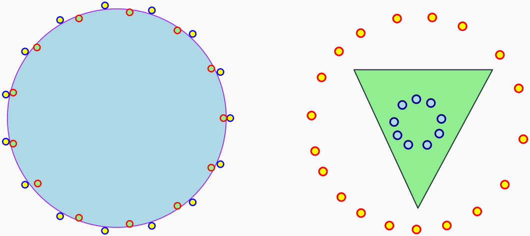

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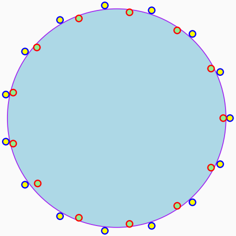
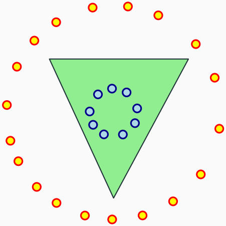
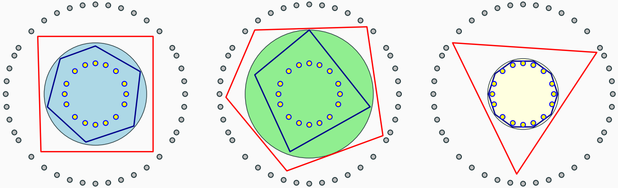
- **Goal:** design instance sensitive algorithms





# A lower bound



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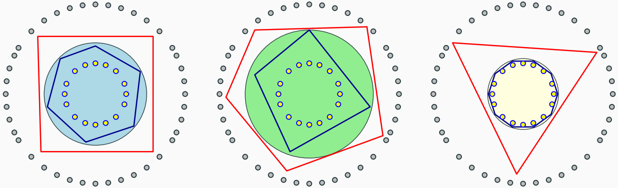

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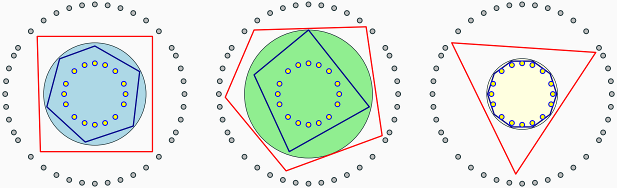

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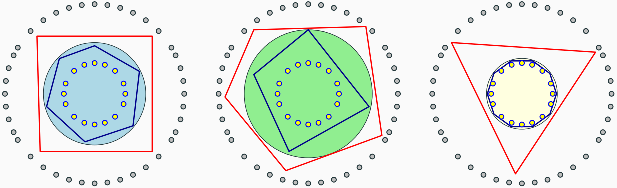

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**Lemma**

Any algorithm must make at least $\sigma(P, C)$ oracle queries.

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| Classify (2D) | $\sigma(P, C)$ | $O(k(P) \log n)$ (†) |
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- $P$ chosen UAR from $[0, 1]^2$
  $\implies \mathbb{E}[k] = \Theta(n^{1/3}) \implies O(n^{1/3} \log n)$

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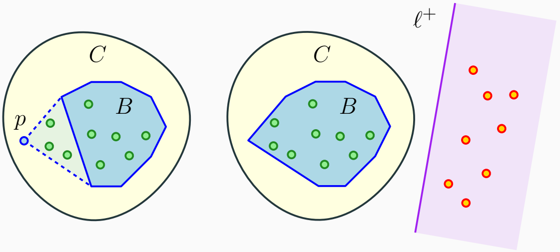

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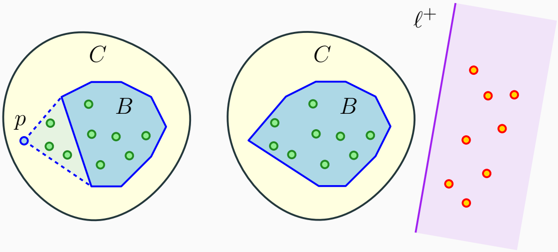
- $c \in \mathbb{R}^2$ is a centerpoint for $P$ if for all halfspaces $\ell^+$: $c \in \ell^+ \implies |P \cap \ell^+| \geq |P|/3$.



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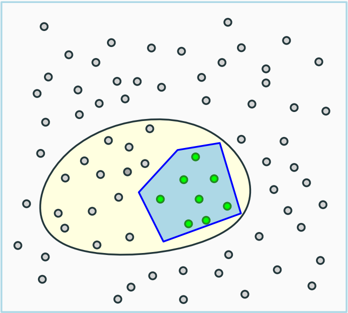

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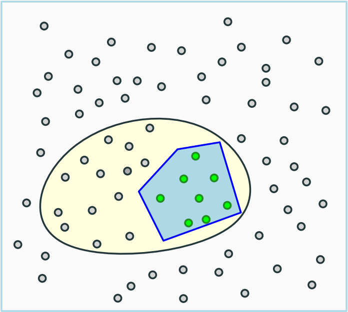

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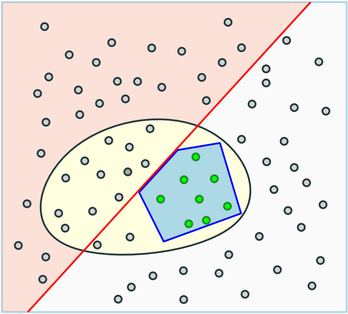

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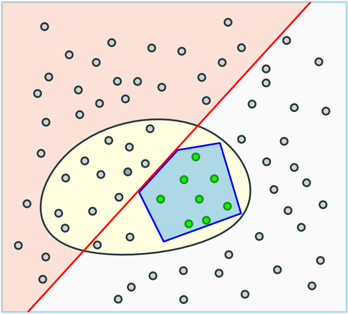

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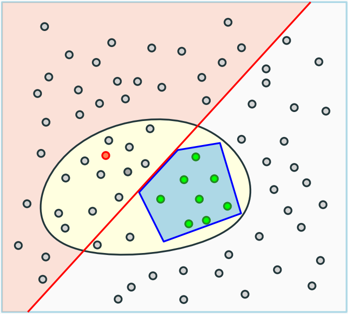

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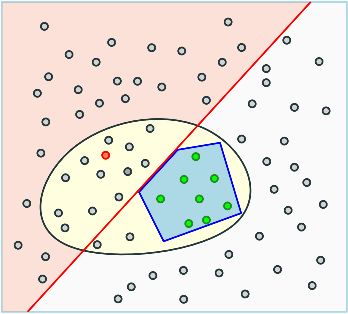

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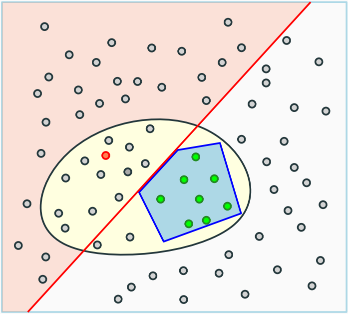

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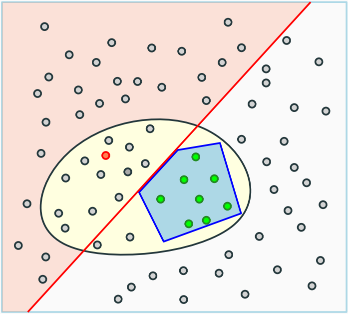

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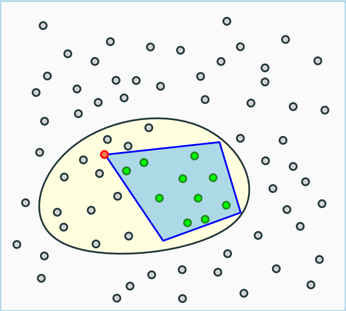
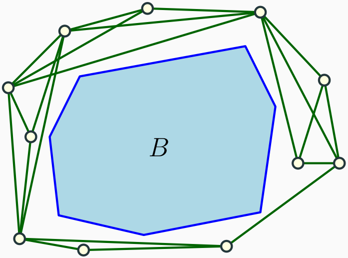


# Animation

# Analysis

- Count visible pairs of points



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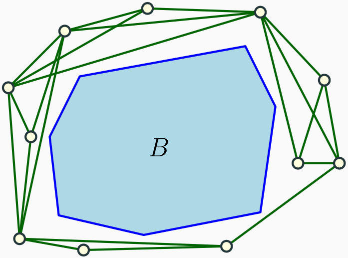

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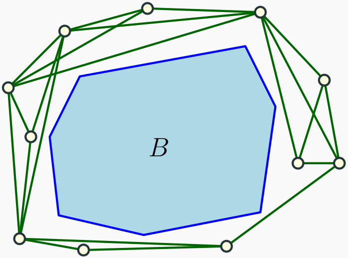

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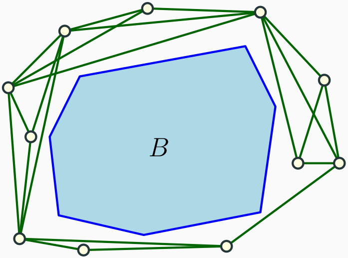

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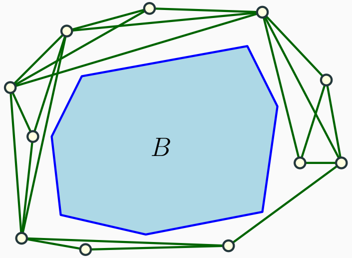

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Greedy algorithm classifies all points using $O(k \log n)$ queries.

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- Shaving log factors?

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| Classify (3D) | — | $O(k(P)\log n)$ |
| Verify in | $\lvert F_{\text{in}}\rvert$ | $O(\lvert F_{\text{in}}\rvert \log n)$ |
| Verify out | $\lvert F_{\text{out}}\rvert$ | $O(\lvert F_{\text{out}}\rvert \log n)$ |

- Shaving log factors?
- Near-optimal solution in 3D?

# Conclusion & open problems

| Problem | Lowerbound | Upperbound |
|---|---|---|
| Classify (2D) | $\sigma(P, C)$ | $O(k(P) \log n)$<br>$O(\sigma(P, C) \log^2 n)$ |
| Classify (3D) | — | $O(k(P) \log n)$ |
| Verify in | $\lvert F_{\text{in}} \rvert$ | $O(\lvert F_{\text{in}} \rvert \log n)$ |
| Verify out | $\lvert F_{\text{out}} \rvert$ | $O(\lvert F_{\text{out}} \rvert \log n)$ |

- Shaving log factors?
- Near-optimal solution in 3D?
- Higher dimensions?

# Conclusion & open problems

| Problem | Lowerbound | Upperbound |
|---|---|---|
| Classify (2D) | $\sigma(P, C)$ | $O(k(P) \log n)$<br>$O(\sigma(P, C) \log^2 n)$ |
| Classify (3D) | — | $O(k(P) \log n)$ |
| Verify in | $\|F_{\mathrm{in}}\|$ | $O(\|F_{\mathrm{in}}\| \log n)$ |
| Verify out | $\|F_{\mathrm{out}}\|$ | $O(\|F_{\mathrm{out}}\| \log n)$ |

- Shaving log factors?
- Near-optimal solution in 3D?
- Higher dimensions?

**Thank you!**

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