Some Geometric Applications of Anti-Chains

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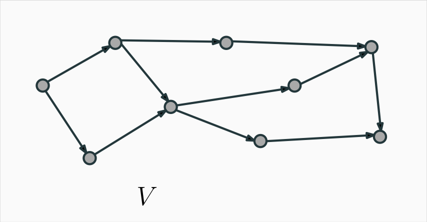
Partial orderings (posets)

- $(V, \prec)$: partially ordered set (poset)



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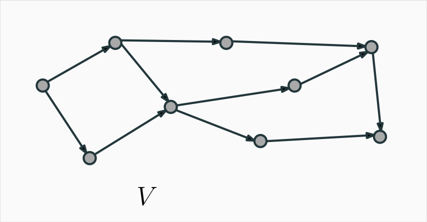

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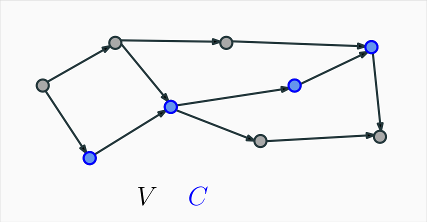

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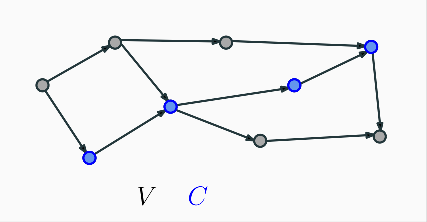

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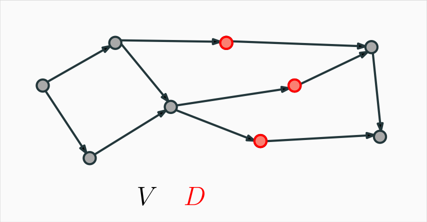

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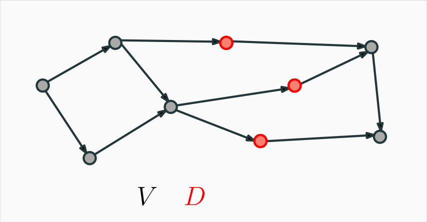

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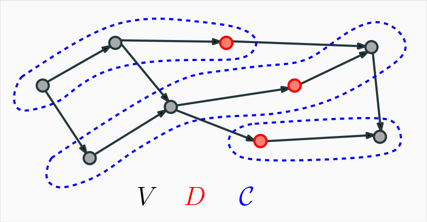

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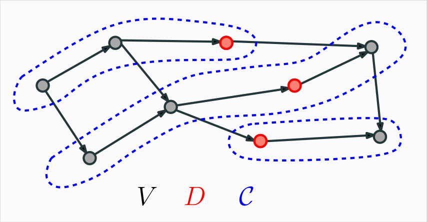
- Largest anti-chain = smallest chain cover [Dilworth, 1950]



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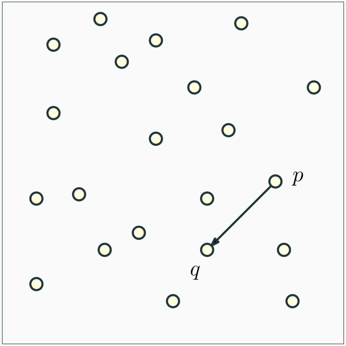

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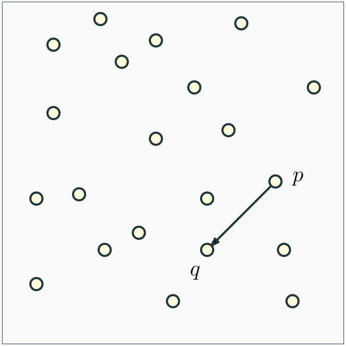

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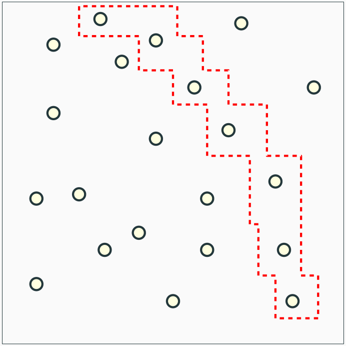

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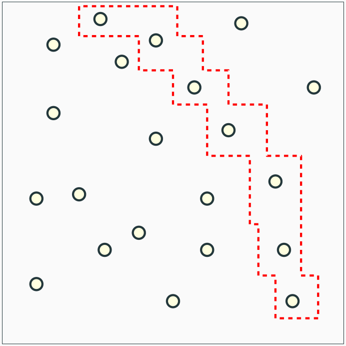
- Quickly find largest Pareto-optimal subset for $(P, \prec)$?



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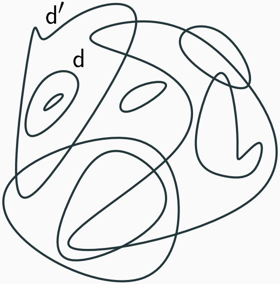

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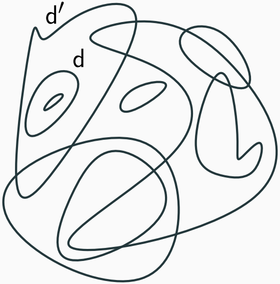

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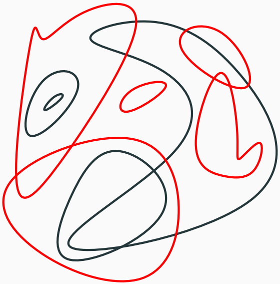

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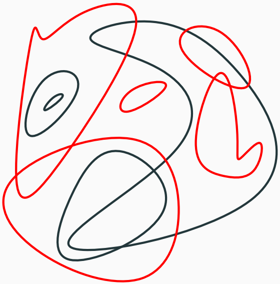

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- Reduces to max matching in bipartite graph G [Hopcroft and Karp, 1973]
- Implicit posets $\implies$ edges of G are implicit
- Possible speedup?

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For $P \subseteq V$, $m = |P|$, we have data structure $\mathscr{D}(P)$:

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Can find largest anti-chain for $(V, \prec)$ in $O\left(n^{1.5} \cdot T(n)\right)$ time

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- Recently: Similar framework for minimum cuts in disk graphs [Cabello and Mulzer, 2020]

Applications

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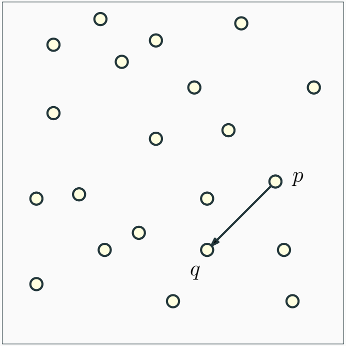

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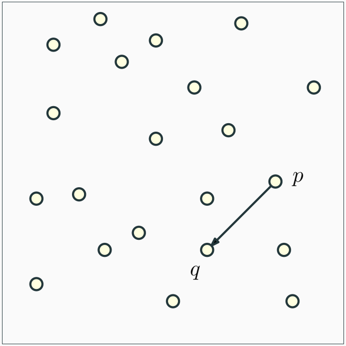
- $Q \subseteq P$ is Pareto-optimal if no point in Q dominates any other point in Q



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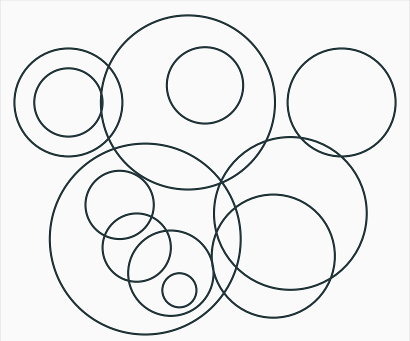
Largest Pareto-optimal subset in $\mathbb{R}^d$ in time $O(n^{1.5}(\log n / \log\log n)^{d-1})$

Largest loose subset of disks

- $\mathcal{D}$: set of disks in $\mathbb{R}^2$



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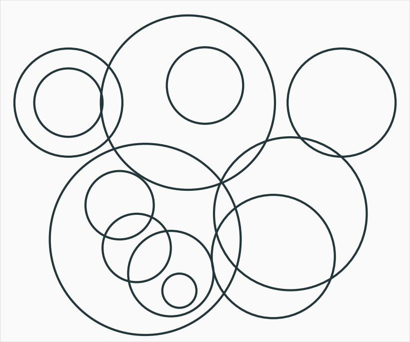

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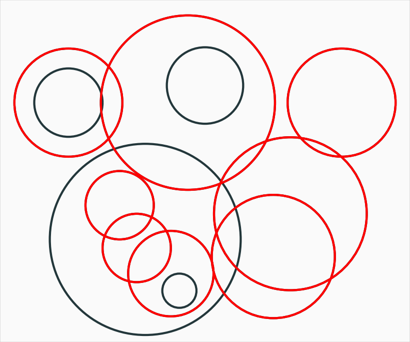

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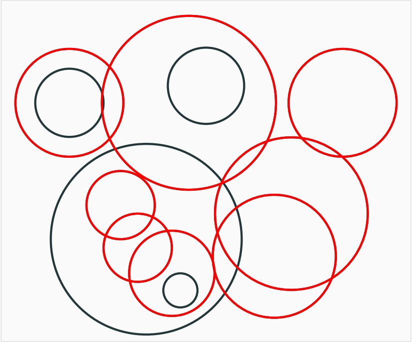

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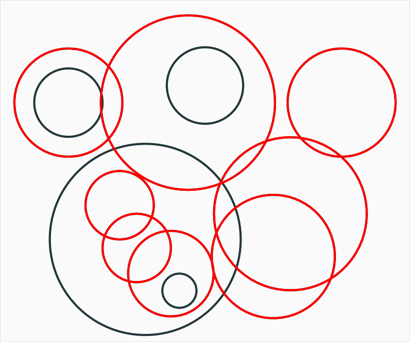
- Compare to Independent set which is NP-hard



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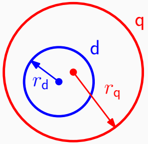
For each $\mathsf{x} \in \mathcal{D}$: $\delta_{\mathsf{x}}(p) = \|c_{\mathsf{x}} - p\| + r_{\mathsf{x}}$.

$\mathsf{d} \subseteq \mathsf{q} \iff \delta_{\mathsf{d}}(c_{\mathsf{q}}) \leq r_{\mathsf{q}}$.



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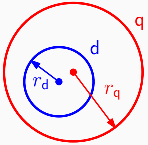

Dynamically maintain $F(p) = \min_{\mathsf{d} \in \mathcal{D}} \delta_{\mathsf{d}}(p)$

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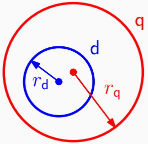

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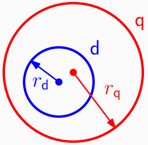
Query $q \in \mathcal{D}$, $\mathscr{D}(\mathcal{D})$ returns $d \in \mathcal{D}$ with $d \subseteq q$

For each $x \in \mathcal{D}$: $\delta_x(p) = \|c_x - p\| + r_x$.
$d \subseteq q \iff \delta_d(c_q) \leq r_q$.



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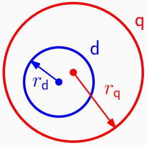
Query $q \in \mathcal{D}$, $\mathscr{D}(\mathcal{D})$ returns $d \in \mathcal{D}$ with $d \subseteq q$

For each $x \in \mathcal{D}$: $\delta_x(p) = \|c_x - p\| + r_x$.
$d \subseteq q \iff \delta_d(c_q) \leq r_q$.



Dynamically maintain $F(p) = \min_{d \in \mathcal{D}} \delta_d(p)$ — lower envelope of surfaces $\{\delta_d \mid d \in \mathcal{D}\}$ in $\mathbb{R}^3$ — in time $O(\log^{10+\varepsilon} n)$ [Kaplan et al., 2017].

Our result

Largest loose subset of disks in $\mathbb{R}^2$ in $O\left(n^{1.5} \log^{10+\varepsilon} n\right)$ time

Conclusion

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Thank you!

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