On Separating Points by Lines

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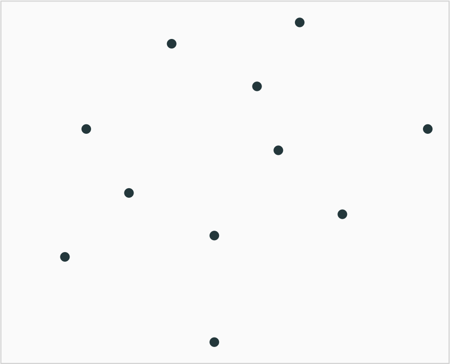
University of Illinois at Urbana-Champaign

Introduction & Motivation

Definition: Separation



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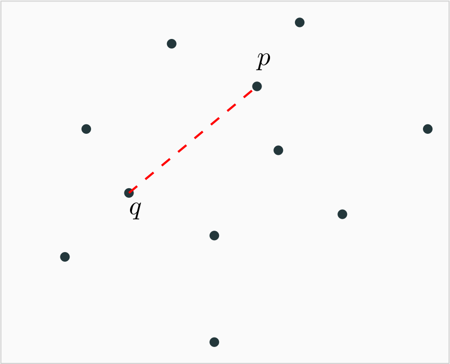

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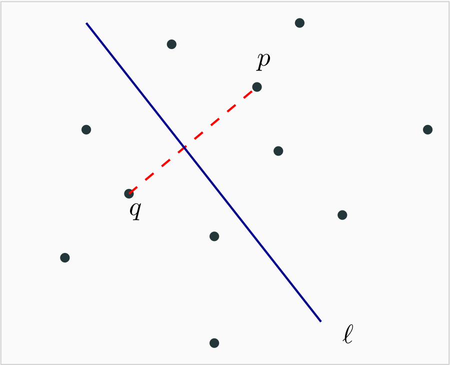

Problem

Let P be a set of points in $\mathbb{R}^2$. Find the minimum number of lines needed to separate all pairs of points in P, denoted by $\mathrm{sep}(P)$.

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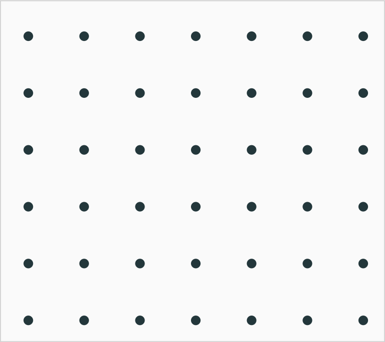
Known: **NP**-complete [Freimer-Mitchell-Piatko '91]

Example: Grid points



$\sqrt{n} \times \sqrt{n}$ grid

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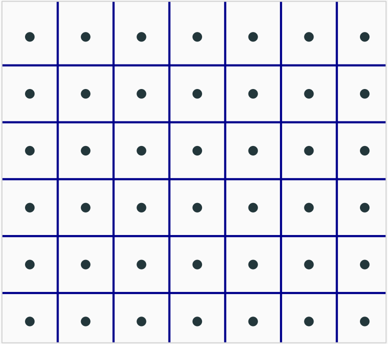
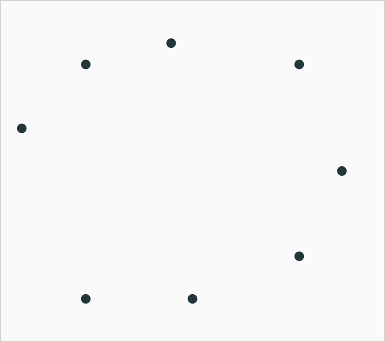


$\sqrt{n} \times \sqrt{n}$ grid $\Rightarrow O(\sqrt{n})$ lines

Example: Convex position



n points in
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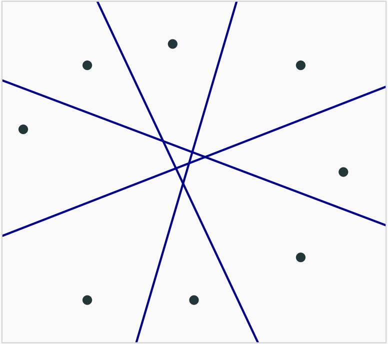
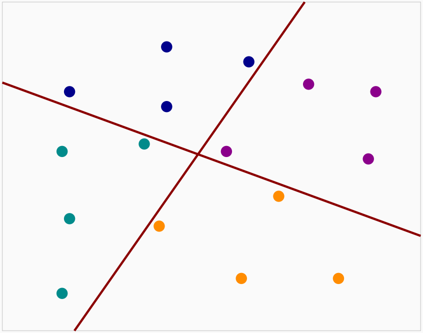


n points in convex position $\Rightarrow n/2$ lines

Motivation

- Ham Sandwich Theorem
- Cutting problems
  [Chazelle-Friedman '90]
- Partition problems
  [Matoušek '92]
- Polynomial partition problems
  [Agarwal-Matoušek-Sharir '13]
  Strong properties, less algorithmically convenient



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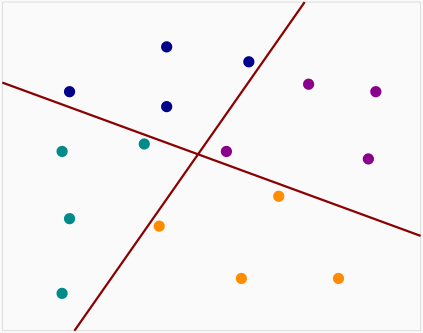
  Strong properties, less algorithmically convenient
- What else can be can with lines/hyperplanes?



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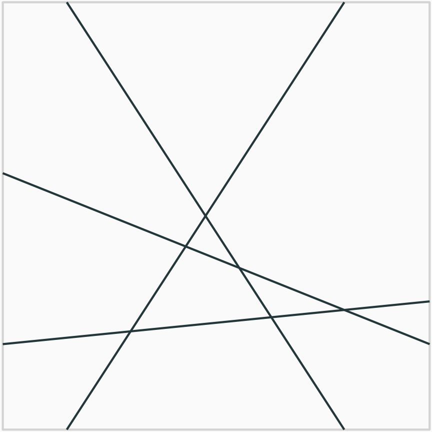

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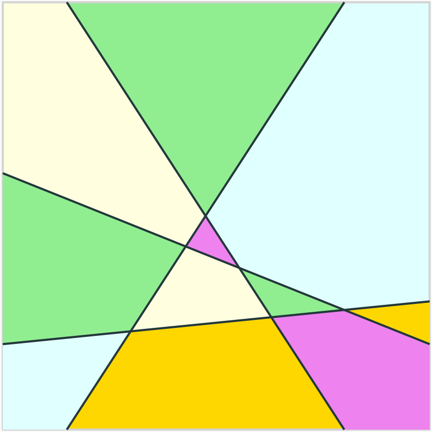
- $O(m^2)$ faces



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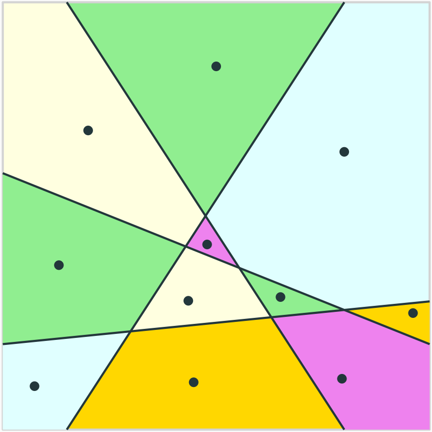

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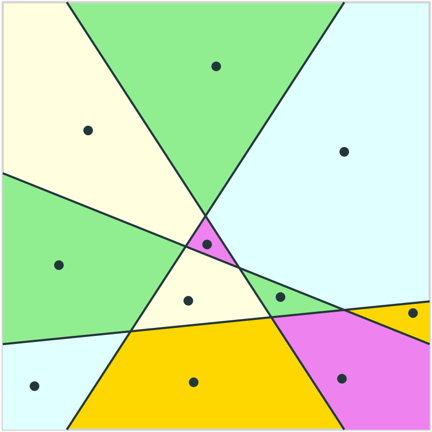


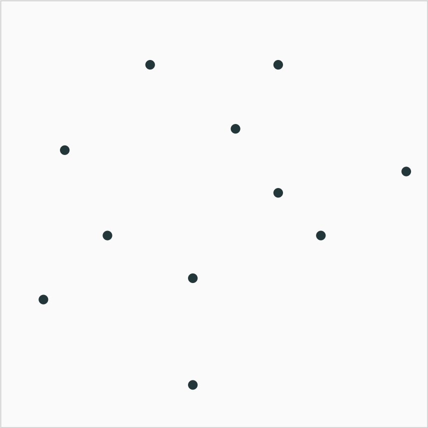
Separating random points

Results

Theorem

Let P be a set of n points chosen UAR from $[0, 1]^2$. With high probability, $\text{sep}(P) = \Omega(n^{2/3} \log \log n / \log n)$.

Expected number of lines: $O(n^{2/3})$



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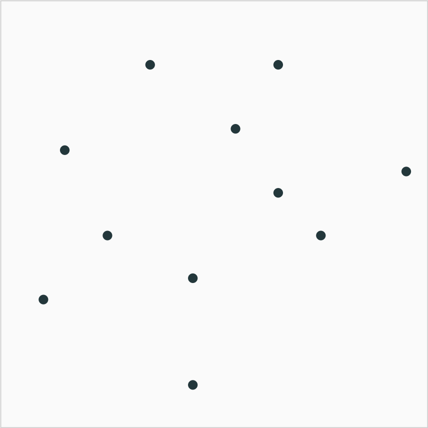

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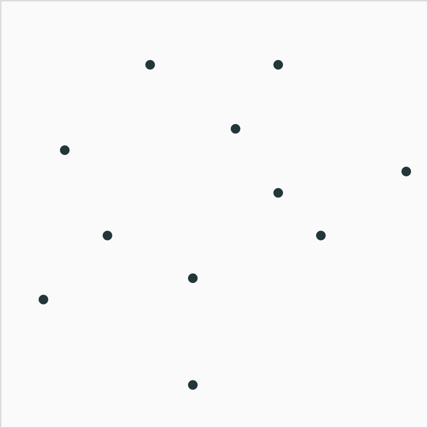

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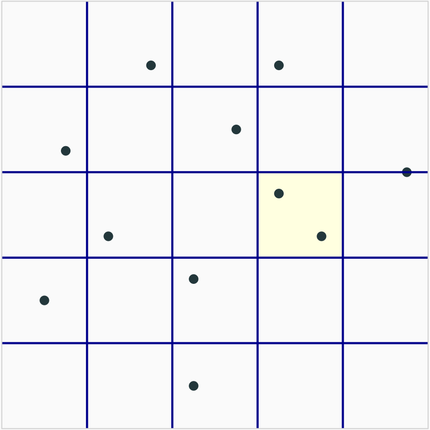

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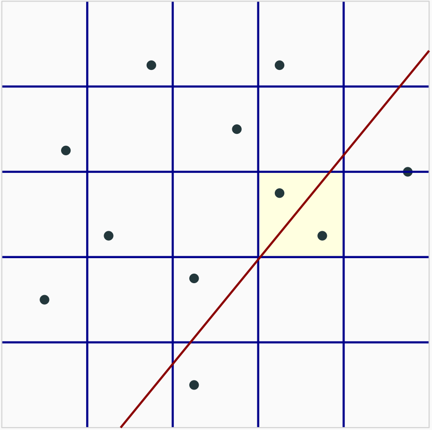

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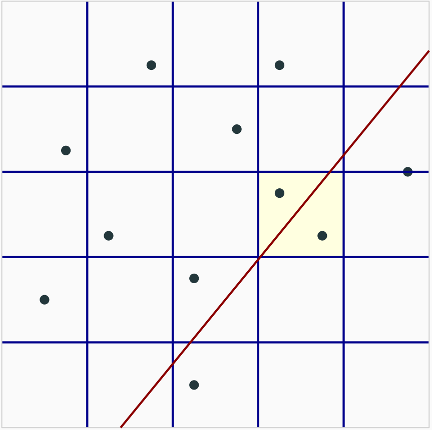

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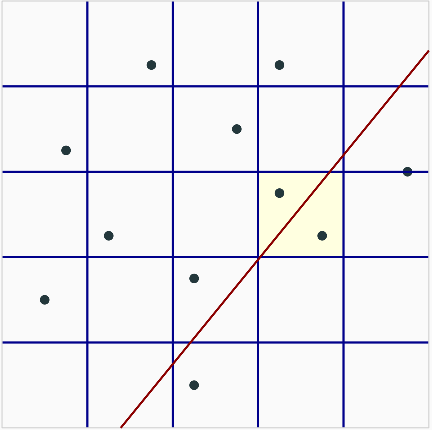

Lower bound (sketch): $\Omega(n^{2/3} \log\log n / \log n)$

Interpret it as a balls (points) and bins (cells) problem on $T \times T$ grid, $T = n^{2/3}$.

1. How many **heavy** cells?
2. How many cells can a line intersect?
3. Of these cells, how many **heavy** cells?

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   $\Longrightarrow \Omega(n^{2/3} \log\log n / \log n)$ lines needed

Higher dimensions

Theorem

Let P be a set of n points chosen UAR from $[0,1]^d$. With high probability, the minimum number of hyperplanes separating P is $\Omega(n^{2/(d+1)} \log\log n / \log n)$.

In expectation, one needs $O(dn^{2/(d+1)})$ separating hyperplanes.

Approximating the minimum separating set

A slightly weaker result

Lemma

Let P be a set of points in $\mathbb{R}^2$ and $\mathrm{OPT} := \mathrm{sep}(P)$.

There is an algorithm that finds set of separating set of lines of size $O(\mathrm{OPT} \log \mathrm{OPT})$, expected running time $O(n^2 \mathrm{OPT} \log \mathrm{OPT})$.

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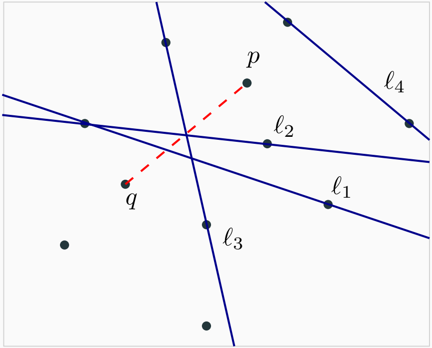
Known: 2-approximation when separating lines are axis-parallel [Calinescu-Dumitrescu-Karloff-Wan '05]

First algorithm with Hitting sets

- Suffices to consider lines passing through pairs of points



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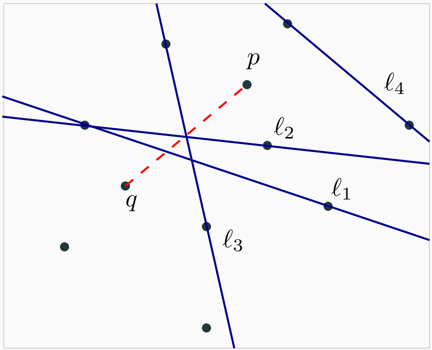

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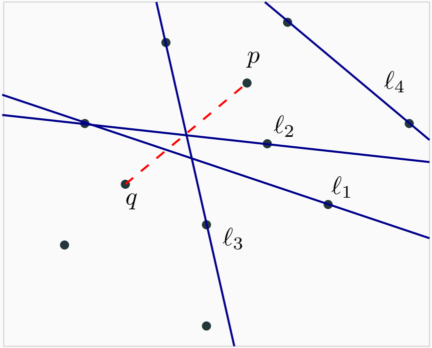

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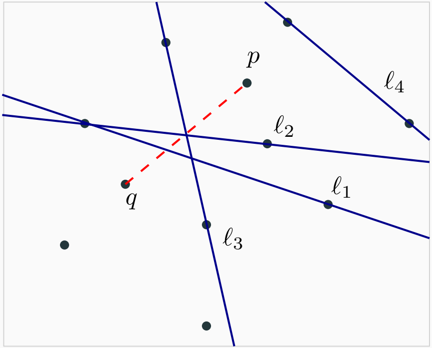

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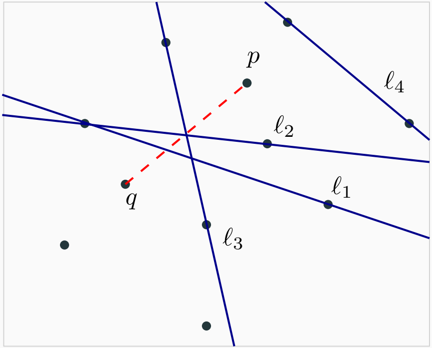

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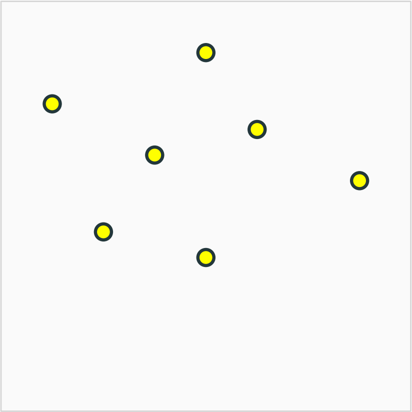

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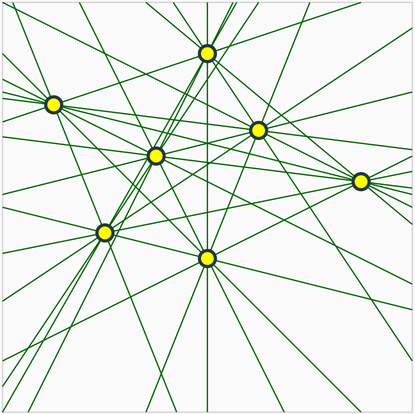

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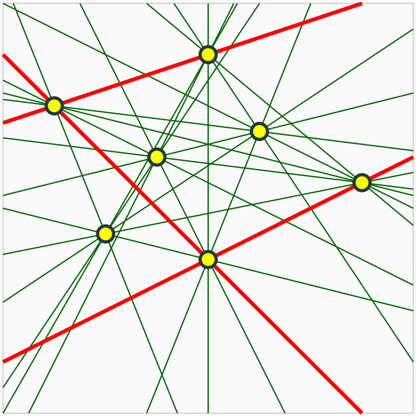

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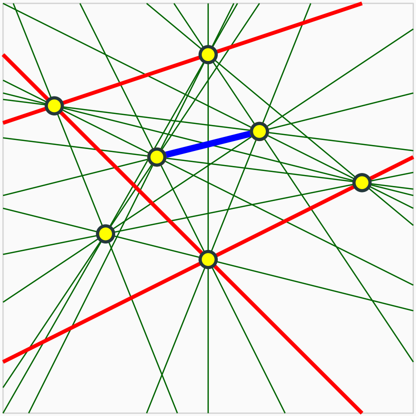

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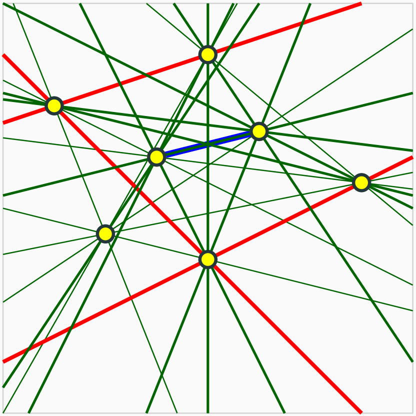

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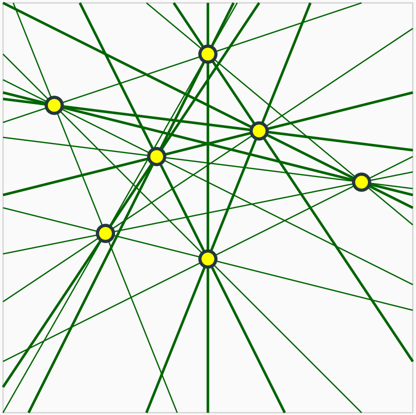

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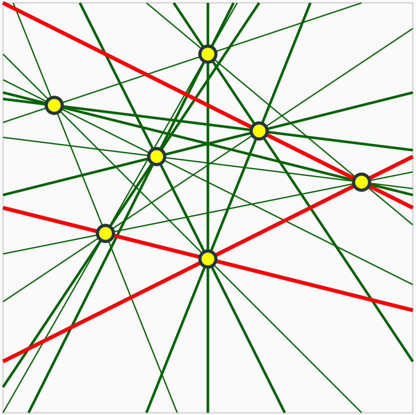

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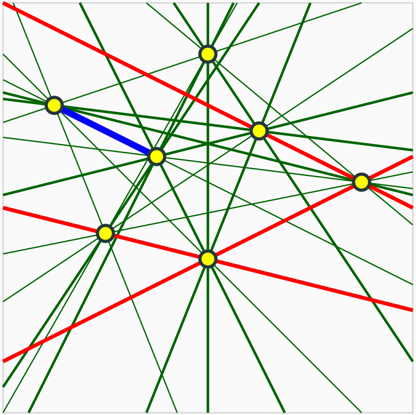

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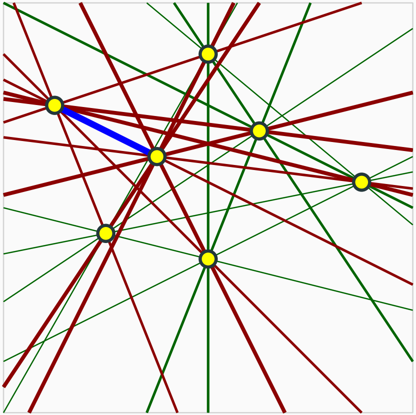

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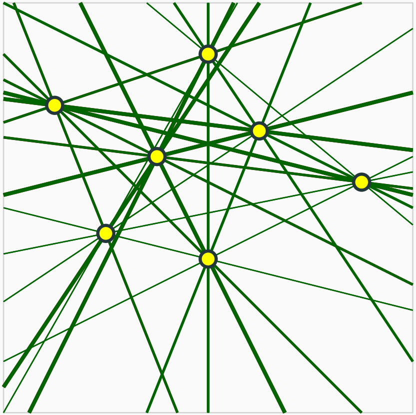

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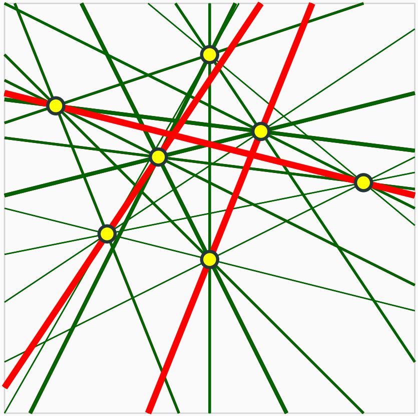


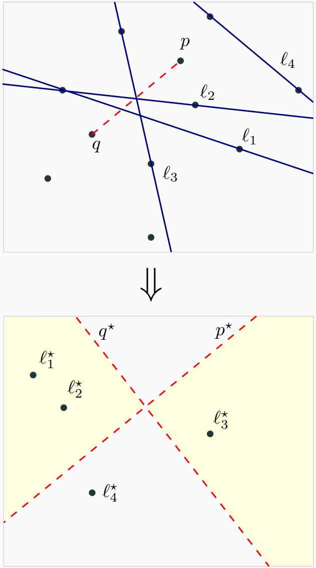
Analysis

Lemma

Algorithm returns a set of separating lines of size $O(\mathrm{OPT} \log \mathrm{OPT})$, expected running time $O(n^2 \mathrm{OPT} \log \mathrm{OPT})$.

Improvement

- Bottleneck is maintaining weights of $O(n^2)$ lines



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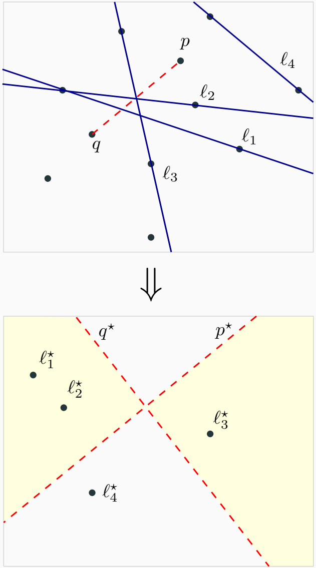
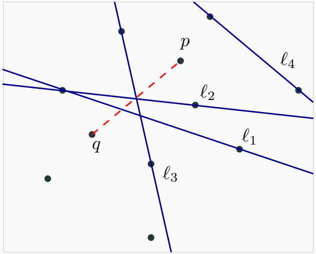
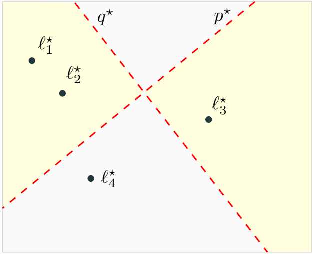


Improvement

- Bottleneck is maintaining weights of $O(n^2)$ lines
- Use duality (lines $\rightarrow$ points, segments $\rightarrow$ wedges)
- Maintain weights as they are updated



$\Downarrow$



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- $\mathrm{sep}(P) = O(\sqrt{n}) \implies O(n^{3/2} \log^{O(1)} n)$
- $\mathrm{sep}(P) = O(n) \implies O(n^{7/3} \log^{O(1)} n)$

Summary

- For n random points in $[0,1]^2$, expected $O(n^{2/3})$ lines needed
- With high probability, $\Omega(n^{2/3} \log\log n / \log n)$ lines needed
- Can compute a separating set of size $O(\mathrm{OPT} \log \mathrm{OPT})$ in time $\widetilde{O}(n^{2/3}\mathrm{OPT}^{5/3})$

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Thank you!