Active Learning a Convex Body in Low Dimensions

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SoCG '19 (YRF), June 18-21, 2019

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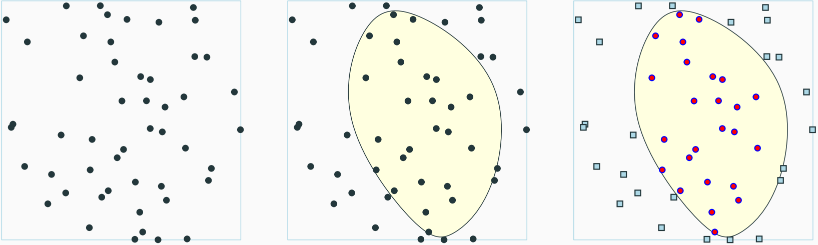
An innocent problem

Problem

Input: $P \subset \mathbb{R}^2$, oracle for unknown convex body C.

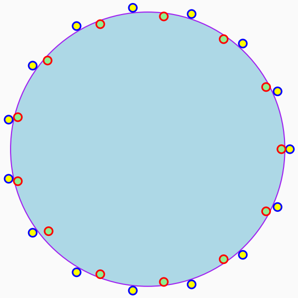
Oracle: Query $q \in \mathbb{R}^2$, returns true $\iff q \in C$.

Goal: Compute $P \cap C$ using fewest number of oracle queries.



Remarks

- Active learning
- Worst case: query all points
- **Question:** In what model can we do better?

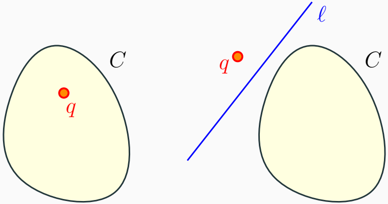


Modified problem

Problem

Input: $P \subset \mathbb{R}^2$, oracle for unknown convex body C.

Oracle: Separation oracle



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Motivation

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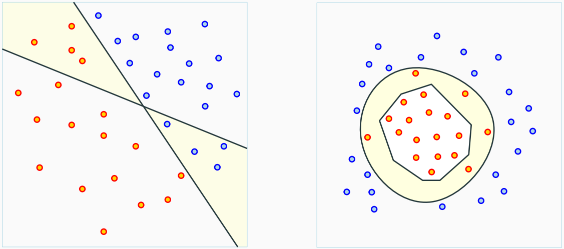
- Other models previously studied [Angluin, 1987] [Panahi, Adler, et al., 2013] [Har-Peled, Kumar, et al., 2016]

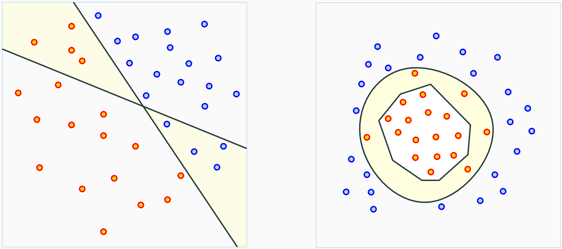
PAC learning

- Allow error in classification
- Random sampling



PAC learning

- Allow error in classification
- Random sampling
- C has bounded complexity $\implies$ finite VC dimension $\implies$ random sample of size $\approx O(\varepsilon^{-1} \log \varepsilon^{-1}) \implies \varepsilon n$ error



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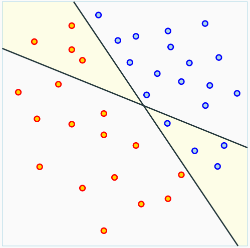
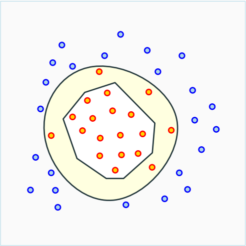
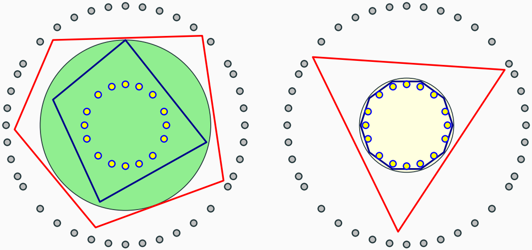
- Scheme fails for arbitrary convex regions





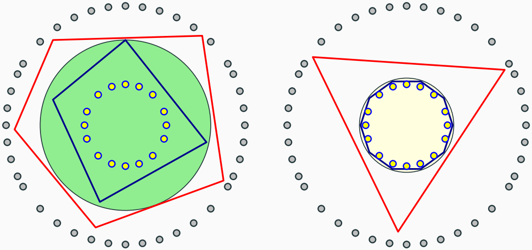
Hard vs. easy instances

- Worst case: query all points

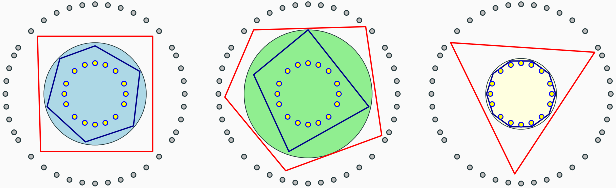


Hard vs. easy instances

- Worst case: query all points
- **Goal:** design instance sensitive algorithms



A lower bound



- F_{in} = convex polygon with fewest vertices s.t. $F_{\text{in}} \subseteq C$ and $C \cap P = F_{\text{in}} \cap P$.

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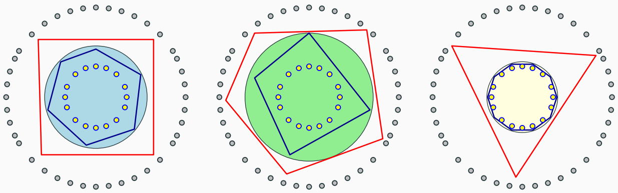

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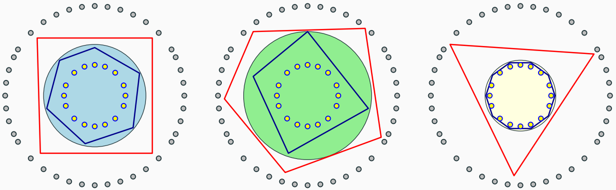

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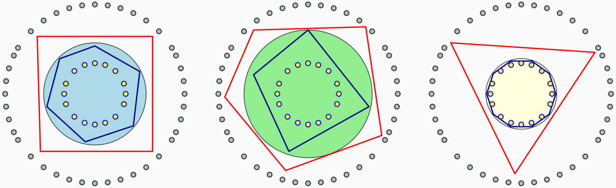

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Lemma

Any algorithm must make at least $\sigma(P, C)$ oracle queries.

Results

Problem	Lowerbound	Upperbound
Classify (2D)	$\sigma(P, C)$	$O(k(P) \log n)$ (†)

(†) $k(P)$ = largest # of pts of P in convex position

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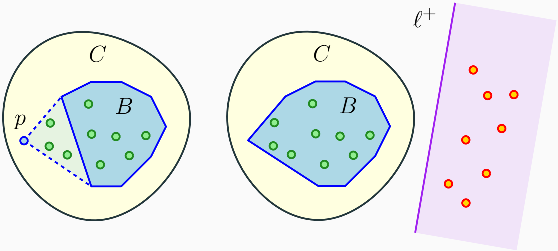

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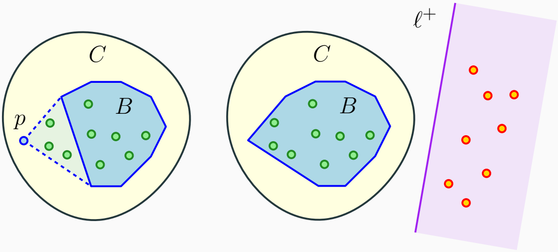
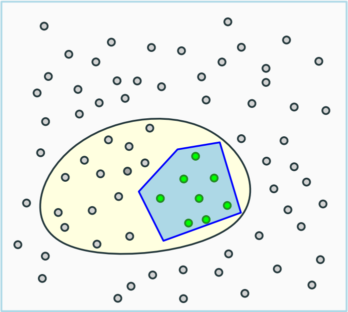
- $c \in \mathbb{R}^2$ is a centerpoint for P if for all halfspaces ℓ^+: $c \in \ell^+ \implies |P \cap \ell^+| \geqslant |P|/3$.



The greedy algorithm

$U \subseteq P$ unclassified points. While $U \neq \varnothing$:



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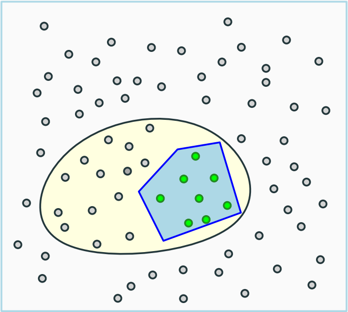
1. $\ell^+ =$ halfspace tangent to B maximizing $|\ell^+ \cap U|$



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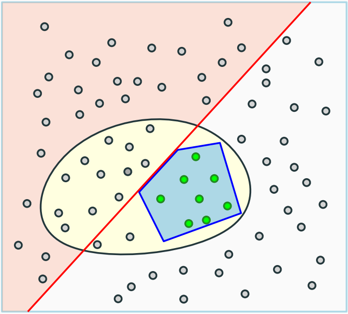

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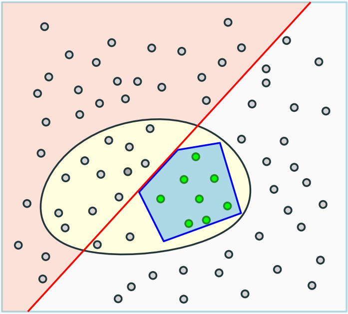

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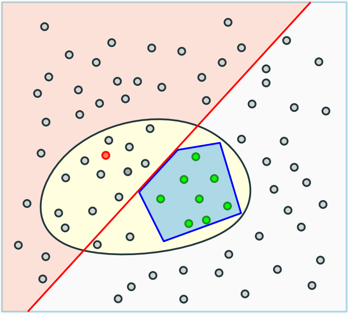

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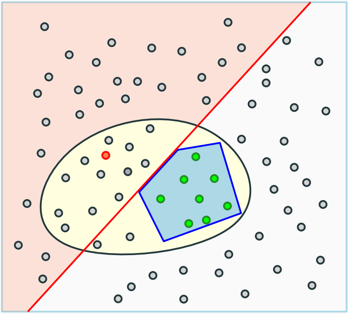

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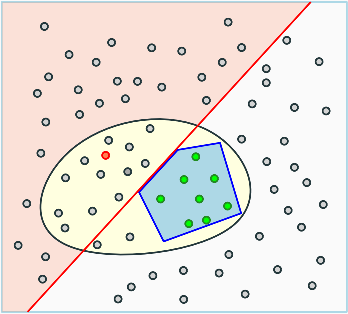
   (A) $c \in C \implies$ **expand**(c)



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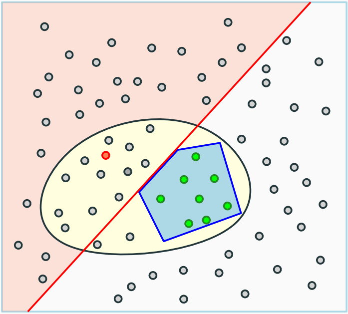

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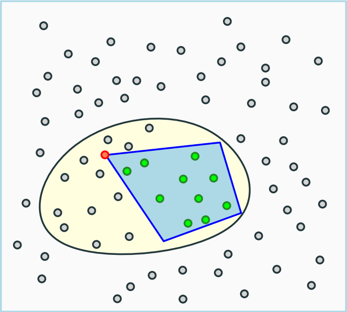
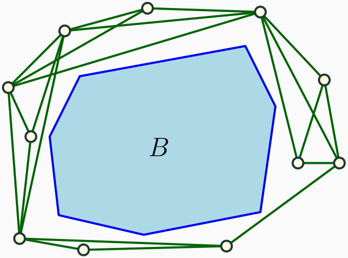


Analysis sketch

- Count visible pairs of points



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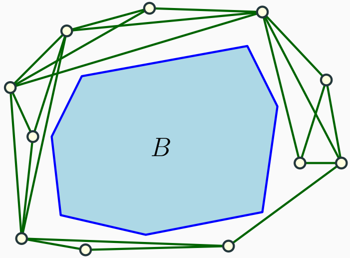

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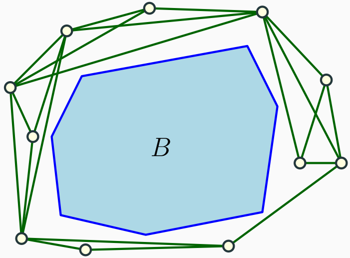

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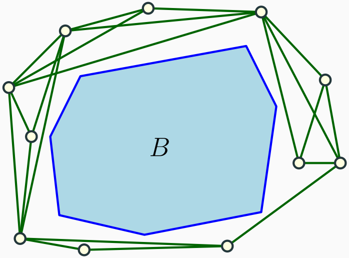

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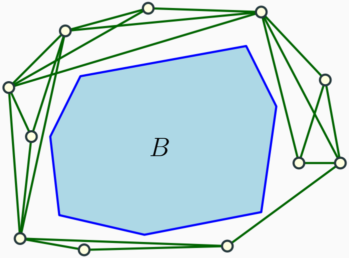

Our result

The greedy algorithm uses $O(k(P) \log n)$ queries.

($k(P) =$ largest # of pts of P in convex position.)

Conclusion & open problems

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- Near-optimal solution in 3D?
- Higher dimensions?
- Conjecture: Greedy extends to $\mathbb{R}^d$ using $O(k(P)^{\lfloor d/2 \rfloor} \log n)$ queries (only interesting when $k(P) \ll (\frac{n}{\log n})^{2/d}$)

References i

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