# Journey to the Center of the Point Set

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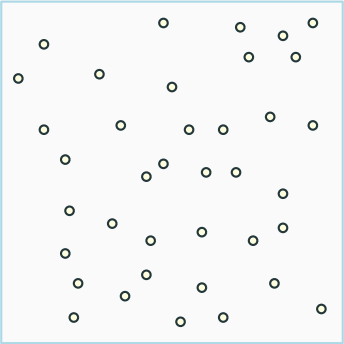
University of Illinois at Urbana-Champaign

Student Presentation

### Definition: Centerpoints

$P \subset \mathbb{R}^d$: set of $n$ points.

$c \in \mathbb{R}^d$ centerpoint for $P$ if for every closed halfspace $h^+$:
$c \in h^+ \implies |P \cap h^+| \geqslant n/(d+1)$.



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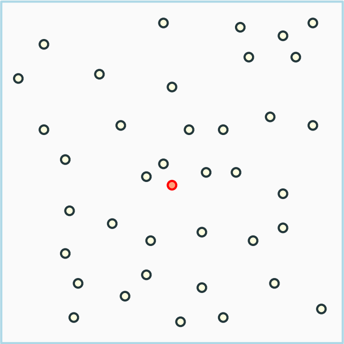

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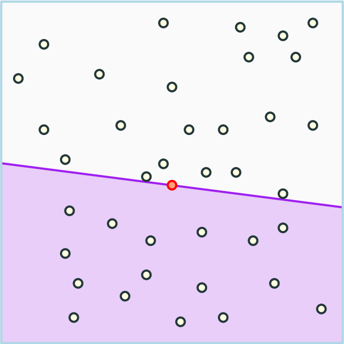

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- $\alpha$-centerpoints for $\alpha \in (0, 1/(d+1)]$

# Previous work: computing centerpoints

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- Can be derandomized [Miller and Sheehy, 2010]
- **Open:** $\approx 1/(d+1)$-centerpoint in $O(\text{poly}(d))$ time?

# A polynomial algorithm

**Theorem** [Clarkson, Eppstein, Miller, Sturtivant, and Teng, 1996]

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# The idea

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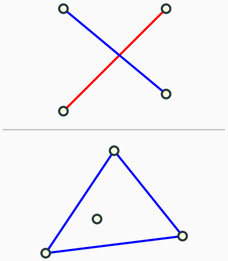
With random sampling, a $\approx 1/(d+2)^2$-centerpoint in time $O(d^7 \log d)$.

- Approximate centerpoint for $d + O(1)$ points in $\mathbb{R}^d$?
- Yes! Radon's Theorem.

### Radon's Theorem

$P \subset \mathbb{R}^d$: set of $d+2$ points.

$\exists$ partition $P = Q \sqcup R$ s.t. $\text{conv}(Q) \cap \text{conv}(R) \neq \varnothing$.



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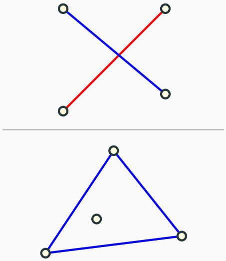

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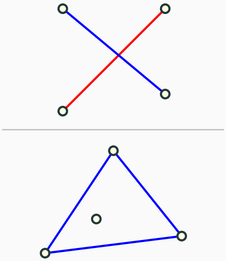


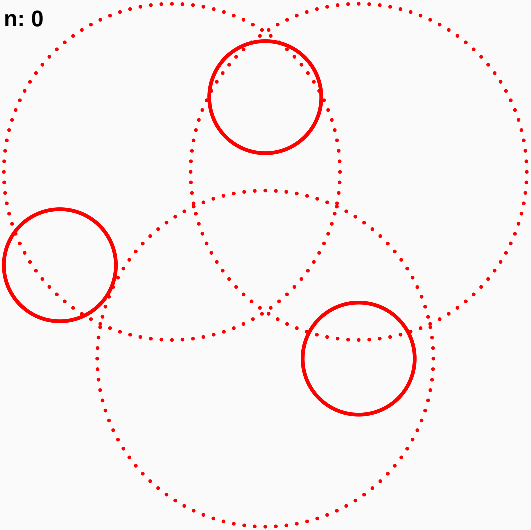
- Radon point: compute in $O(d^3)$ time.
- Radon point: $2/(d + 2)$-centerpoint for $P$.

# Our contribution

A simplified variant of [Clarkson, Eppstein, et al., 1996].

1. $Q \subseteq P$ sample of size $\approx O(d^3 \log d)$ [Li, Long, et al., 2001]
2. For $i = 1, \ldots, O(d|Q|)$:
   - 2.1 Sample $d + 2$ points of $Q$
   - 2.2 Compute their radon point $r$
   - 2.3 Add $r$ to $Q$
   - 2.4 Delete a random point from $Q$ (which isn't $r$)

# Visualization



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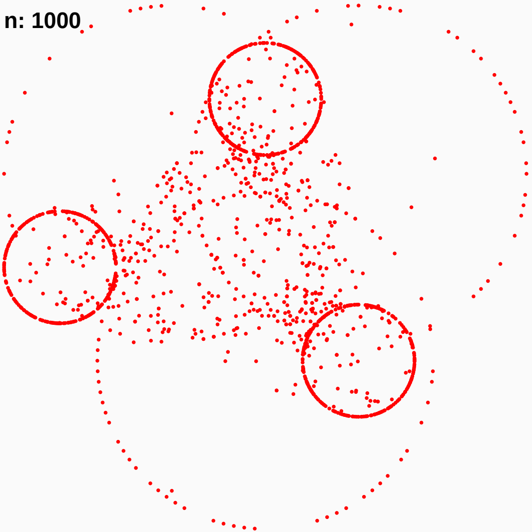
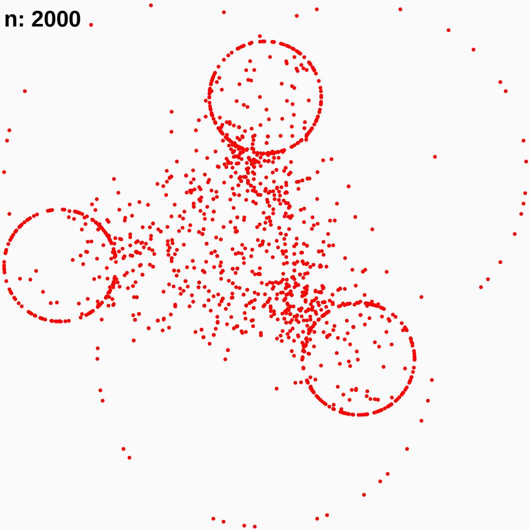
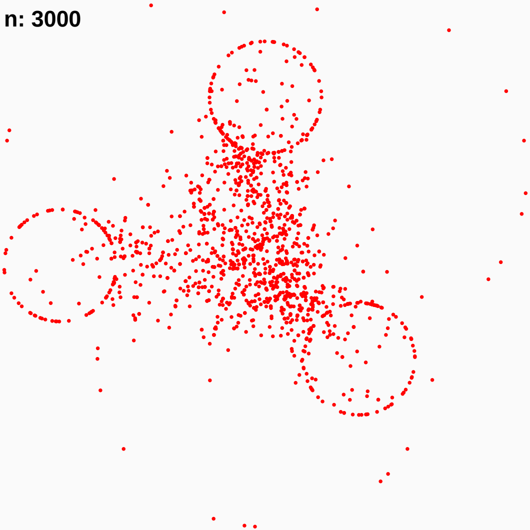



n: 2000

# Visualization



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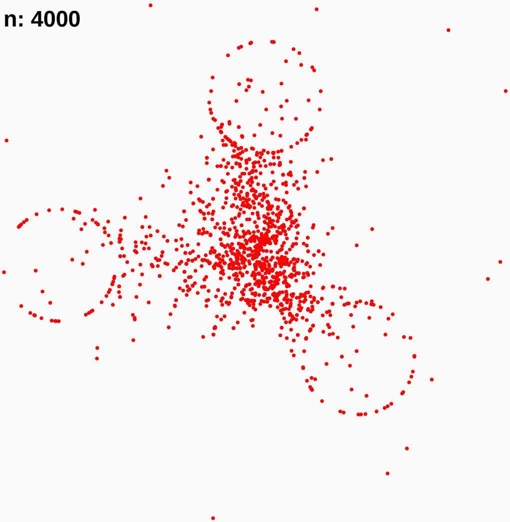

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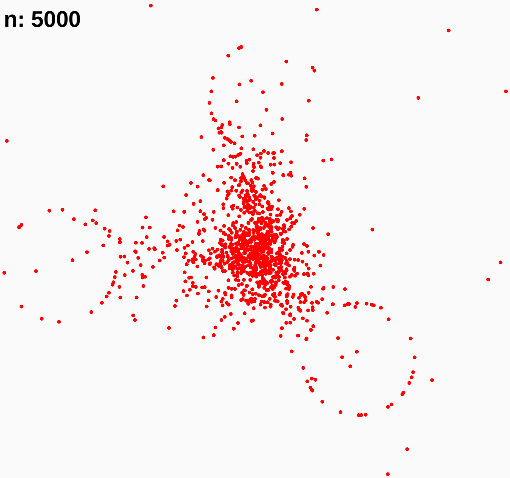

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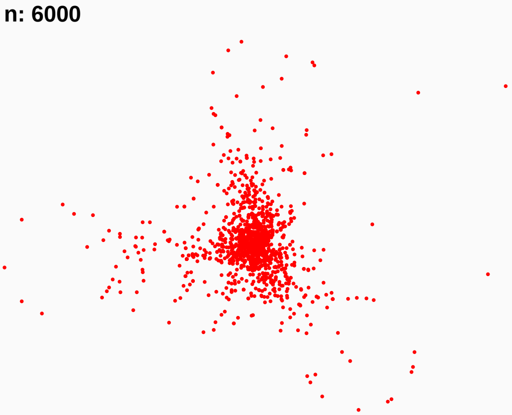
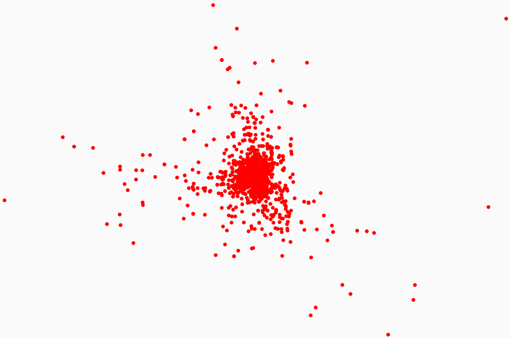
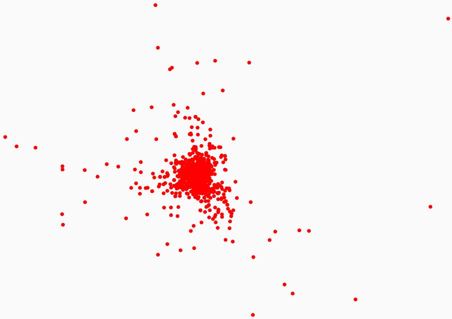


n: 7000



# Visualization

**n: 8000**



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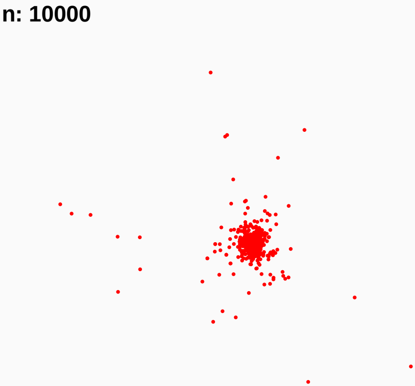

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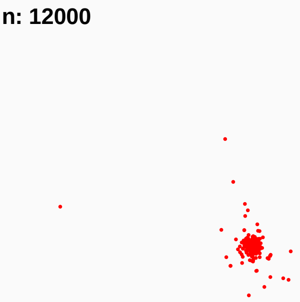
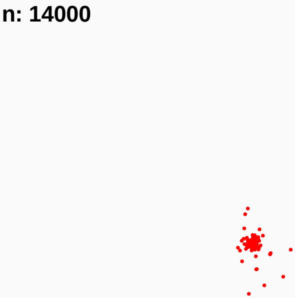



n: 14000

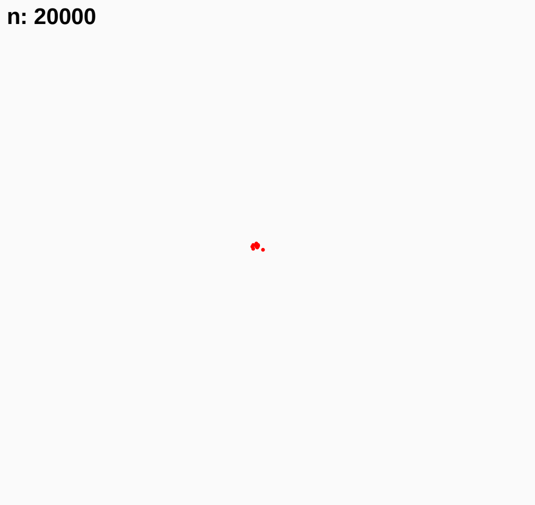
# Visualization

**n: 16000**

# Visualization

n: 18000

# Visualization



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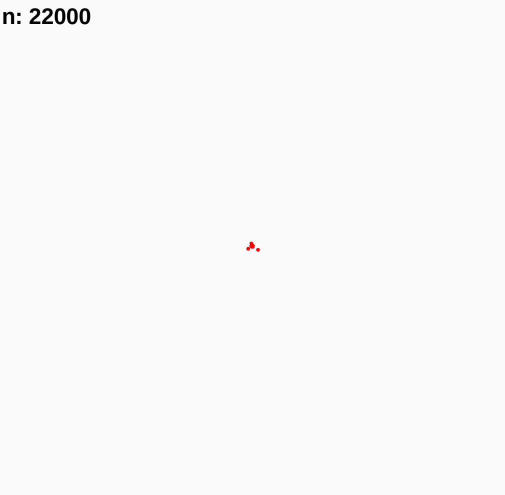

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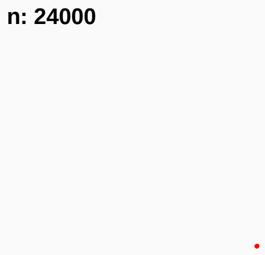
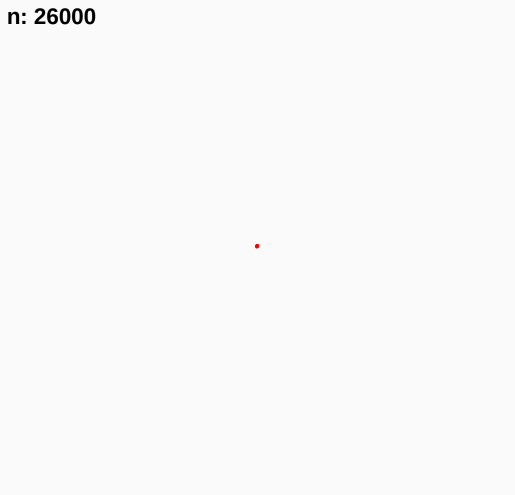


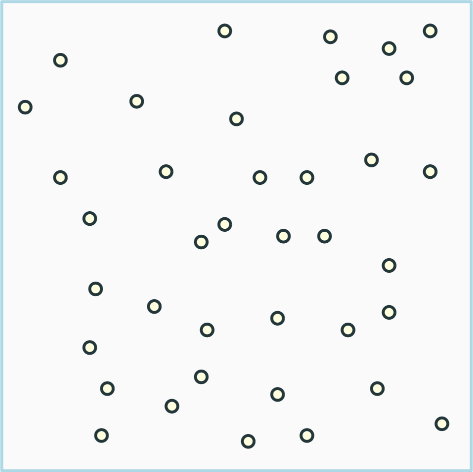

n: 26000

# Visualization

**n: 28000**

# Why does this work?

$q$ is an $\alpha$-centerpoint for $P$ $\iff$ all halfspaces $h^+$ with $|P \cap h^+| > (1-\alpha)|P|$ contain $q$



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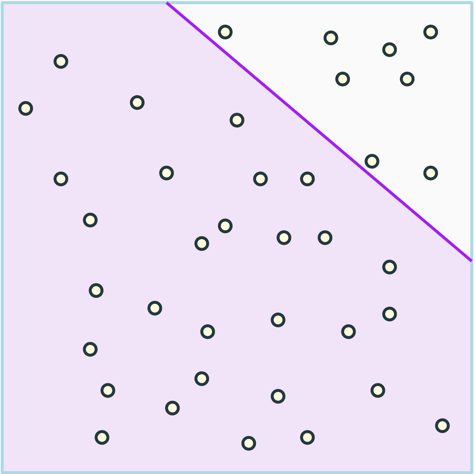

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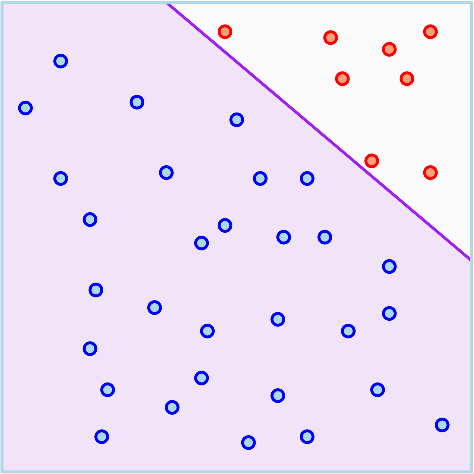

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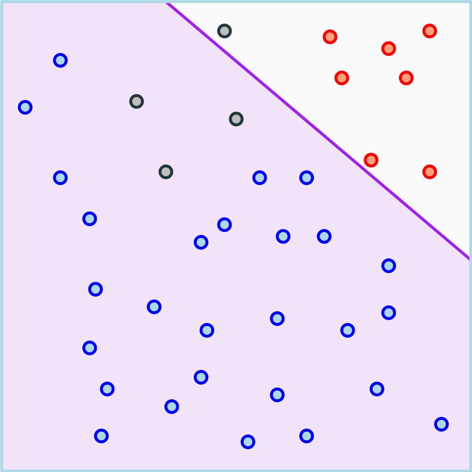

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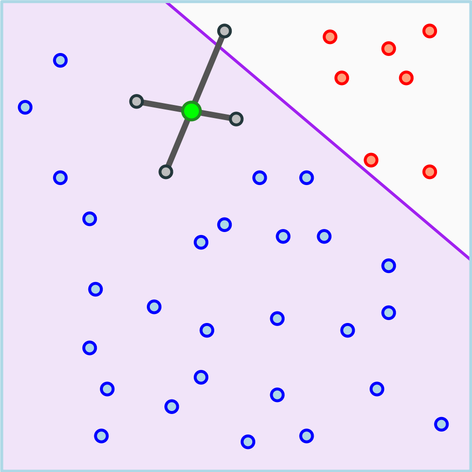

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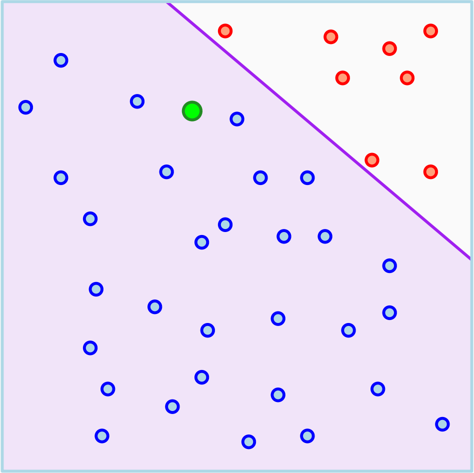

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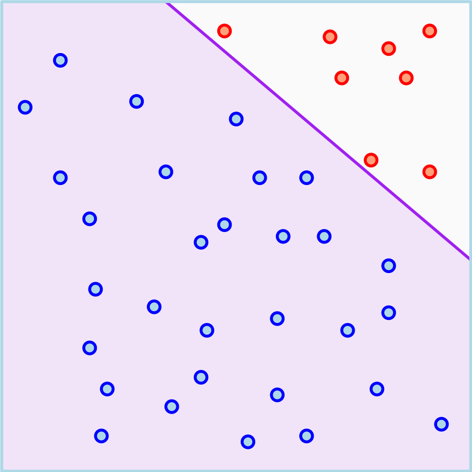

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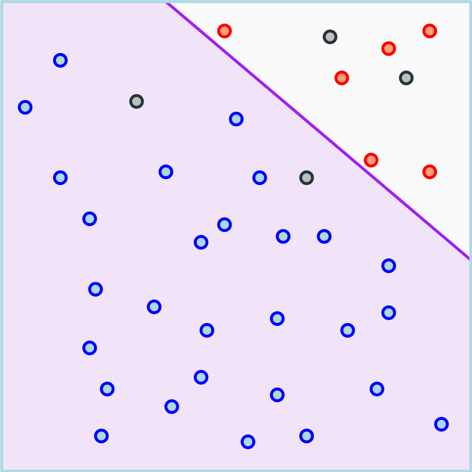

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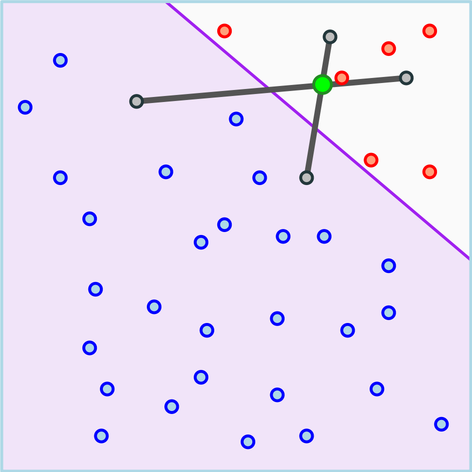
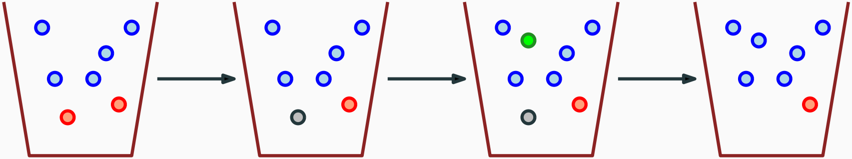


# Radon's urn

- Urn with $b$ blue balls, $r = m - b$ red balls.



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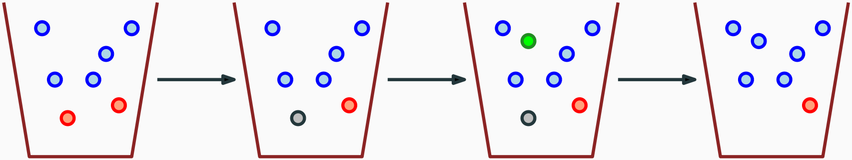

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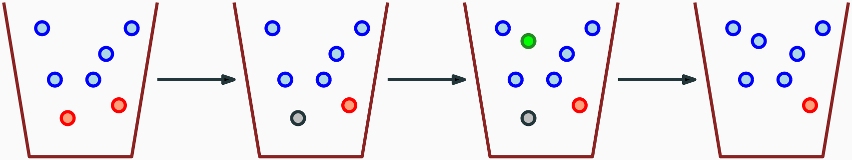

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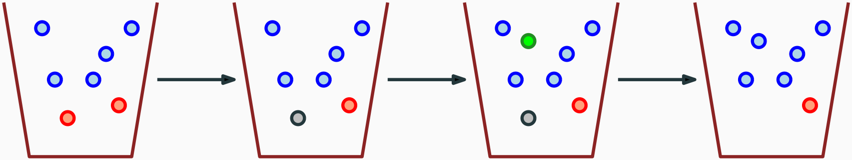

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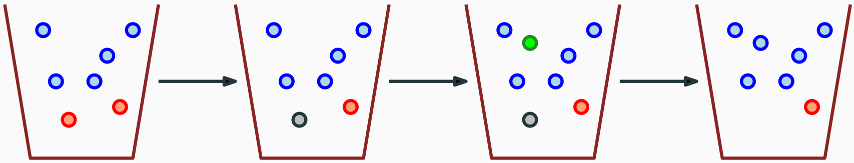

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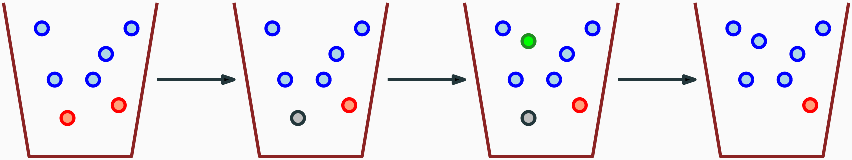
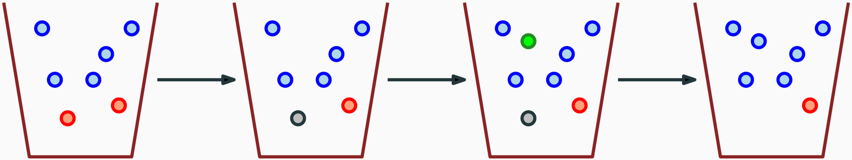
  4. Remove marked ball from urn.



Random walk process



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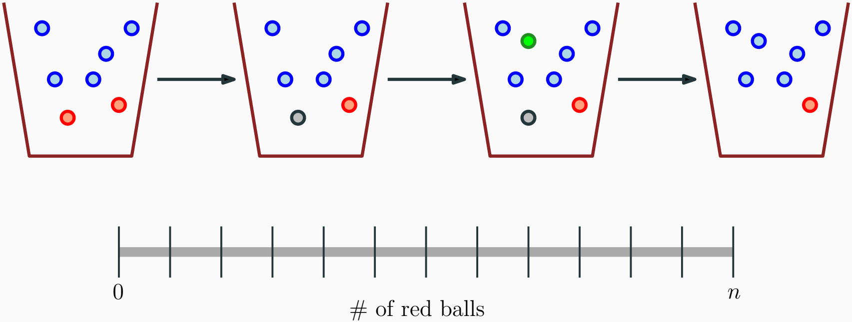

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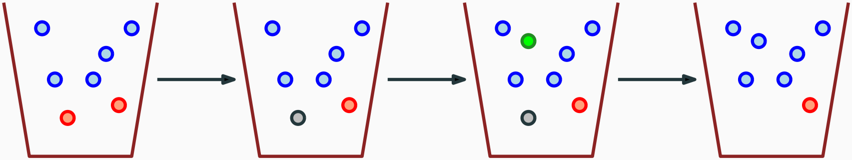

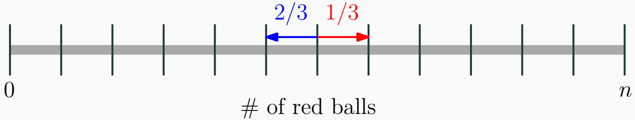

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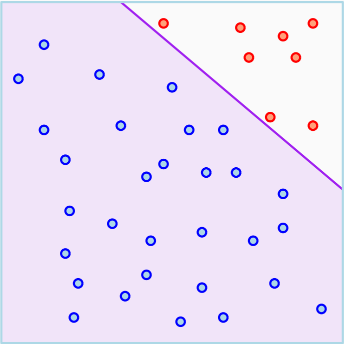
**Problem**

Number of rounds until all balls are blue?



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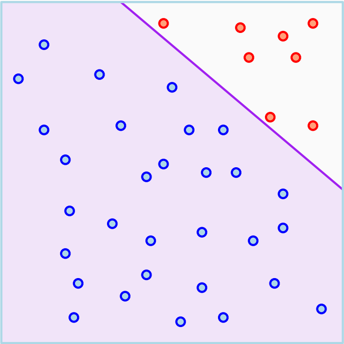
Number of rounds until all balls are blue?

### Our result

When # of balls $m$ is sufficiently large: $O(m \log^2 m)$ rounds.



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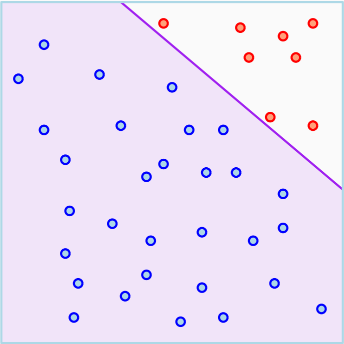
- Simulate random walk process in parallel for all $O(n^d)$ halfspaces.



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**Problem**

$\approx 1/d$-centerpoints in $O(\text{poly}(d))$ time?

### Definition: center nets

$P \subset \mathbb{R}^d$: set of $n$ points.

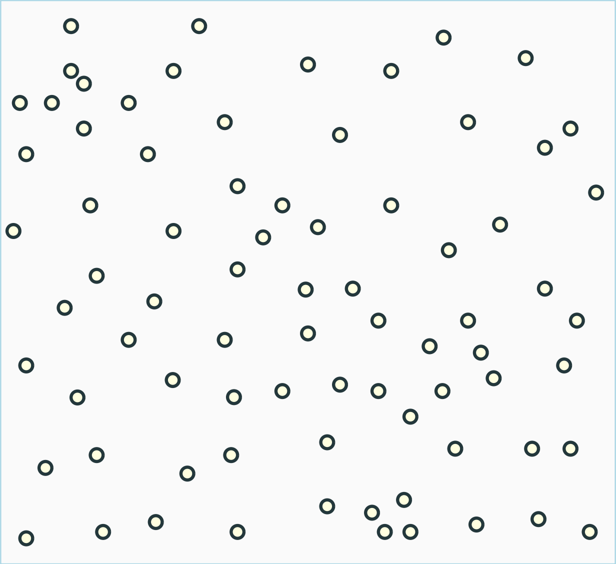
$Q \subset \mathbb{R}^d$, $(\varepsilon, \alpha)$-center net if $\forall$ convex bodies $C \subseteq \mathbb{R}^d$:

$|P \cap C| \geqslant \varepsilon n \implies \exists q \in Q \cap C$, $q$ an $\alpha$-centerpoint of $P \cap C$.

### Our result

There exists an $\left(\varepsilon, \Omega\left(\frac{1}{d \log \varepsilon^{-1}}\right)\right)$-center net for $P$ of size $\widetilde{O}\left((d^2/\varepsilon)^{d^2}\right)$.

# Application: Functional nets



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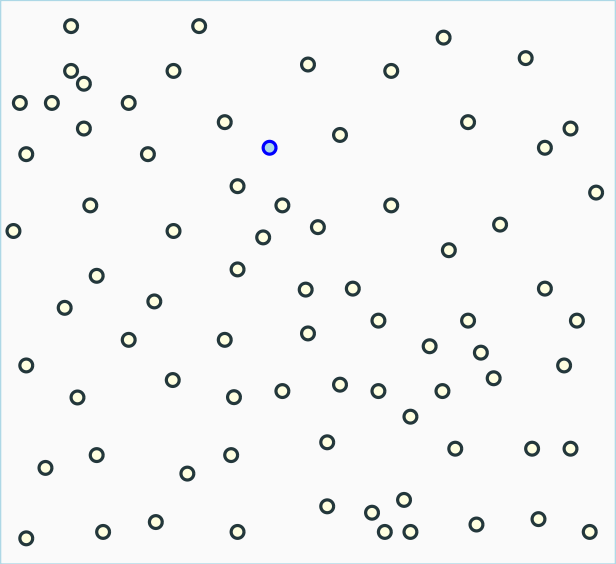

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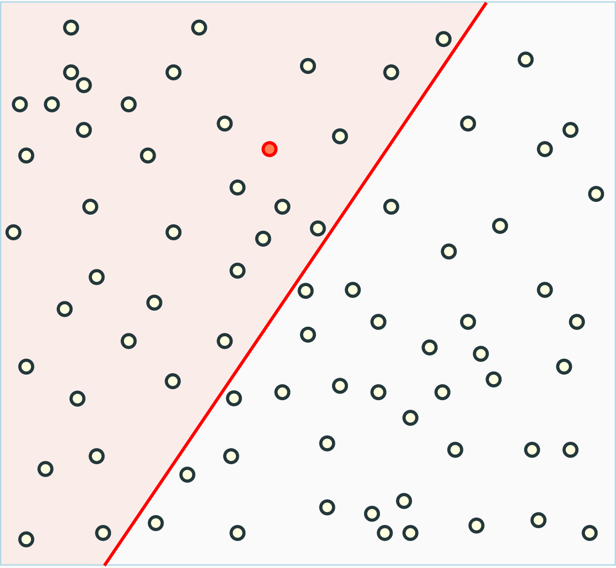

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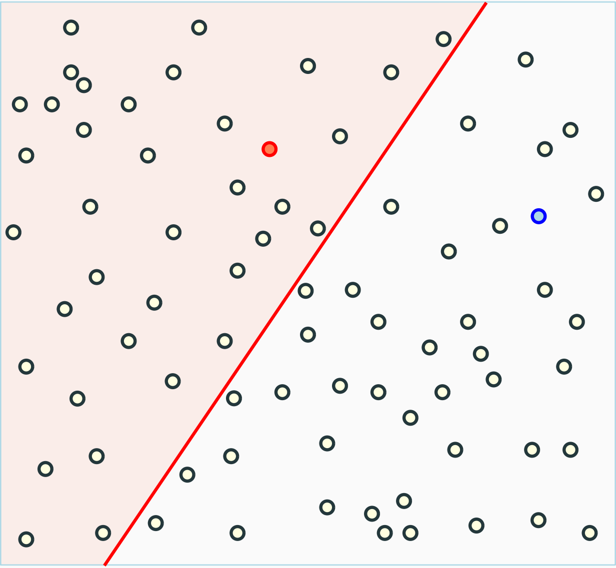

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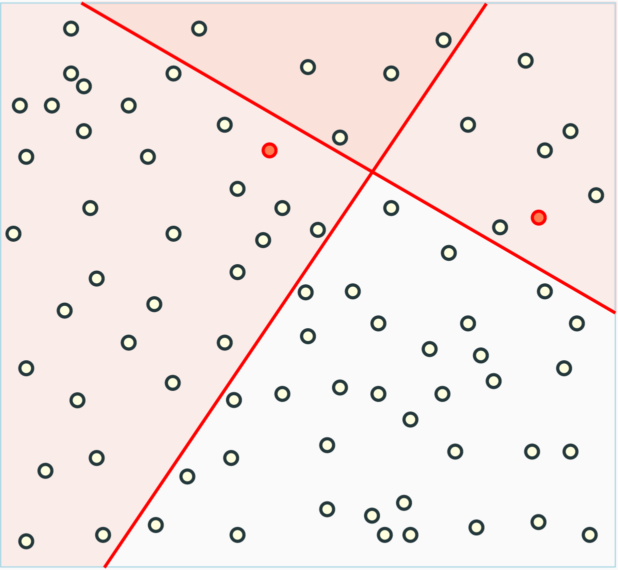

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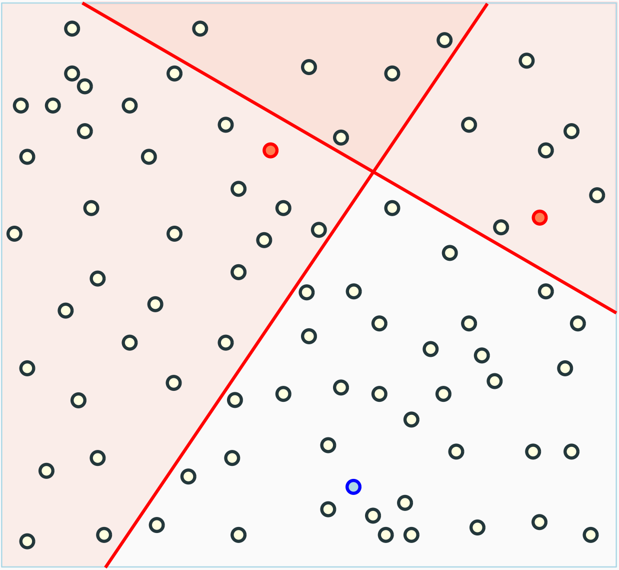

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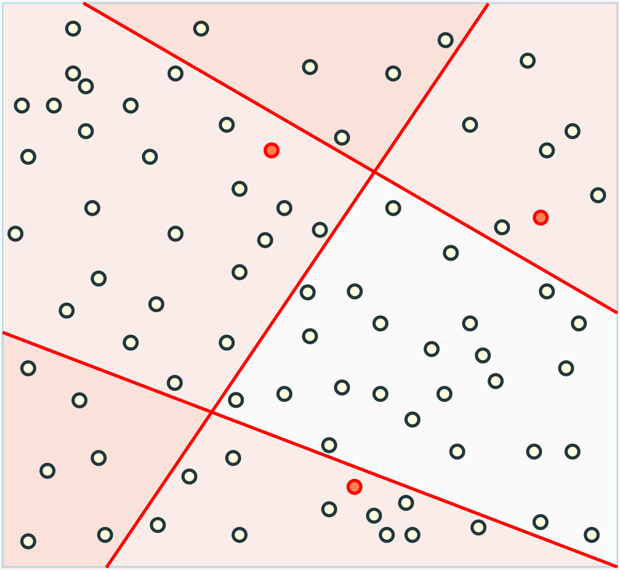

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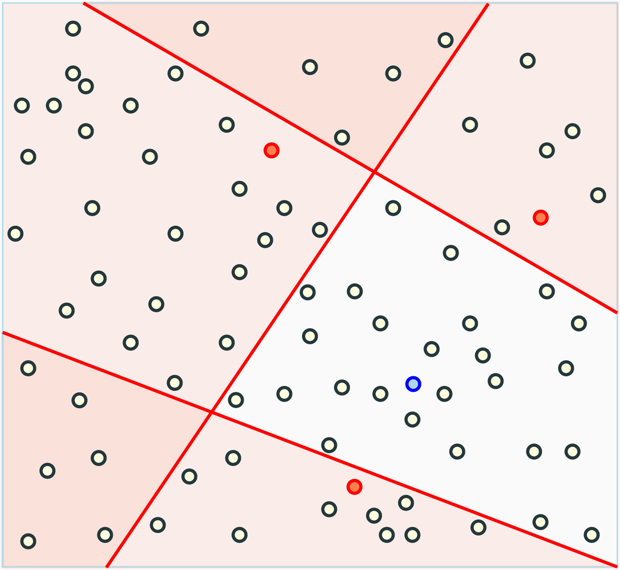

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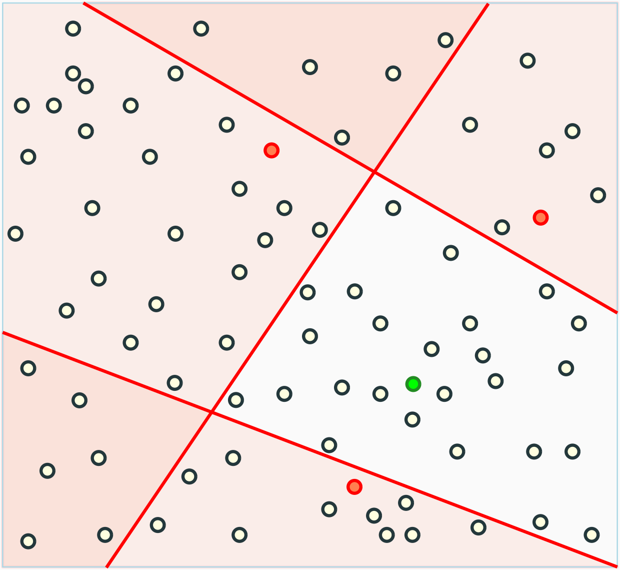

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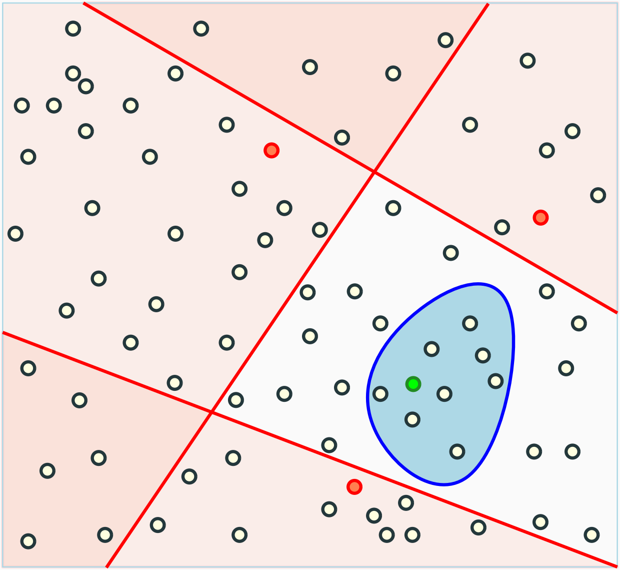

# Application: Functional nets

**Our result**

Can verify if $|C \cap P| \leqslant \varepsilon n$ with $O(d^2 \log \varepsilon^{-1})$ oracle queries to $C$, in $\widetilde{O}(d^9/\varepsilon)$ randomized time.

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- Weak $\varepsilon$-nets in a different model
- Weak $\varepsilon$-nets have exponential dependency on $d$ [Matoušek and Wagner, 2004] [Mustafa and Ray, 2008]
- What models can we obtain similar results with better dependency on $d$?

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