Locality-Sensitive Orderings & Their Applications

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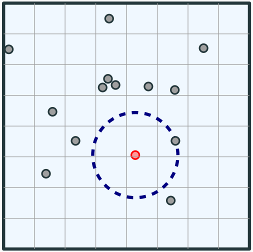
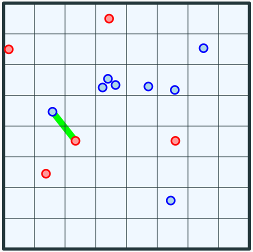
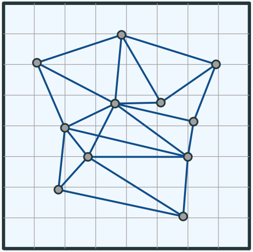
Low dimension proximity problems: $d = O(1)$



Nearest neighbor



Closest pair problems



Spanners/MST

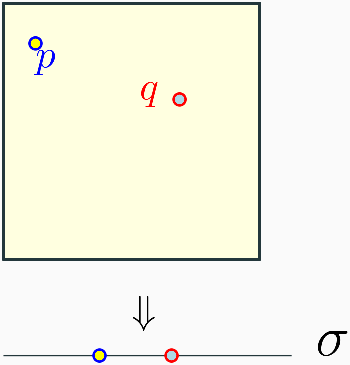
Goal: Dynamic data structures which maintain/return a $(1 + \varepsilon)$-approximation

In this talk

- Quadtrees: Basic data structure in computational geometry
- Many orderings of points in $\mathbb{R}^d$ ($\mathbb{Z}$-order)
- Two new tricks to the mix

  $\Longrightarrow$ Simpler data structures for many proximity problems (plus some new results)

New technique: Locality-sensitive orderings



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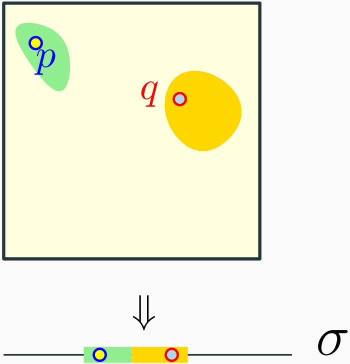

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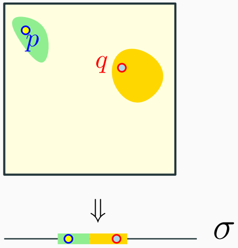

Definition: Locality-Sensitive Orderings

Let $\varepsilon \in (0, 1)$. A collection of orderings Π over $[0, 1)^d$ s.t. for all $p, q \in [0, 1)^d$, exists $\sigma \in \Pi$ where:

$$\forall p \prec_\sigma z \prec_\sigma q : \min(\|z - p\|, \|z - q\|) \leqslant \varepsilon \|p - q\|.$$

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Theorem

There are locality-sensitive orderings of size $O\big((1/\varepsilon^d)\log(1/\varepsilon)\big)$.

Main applications

- New: $(1+\varepsilon)$-bichromatic closest pair

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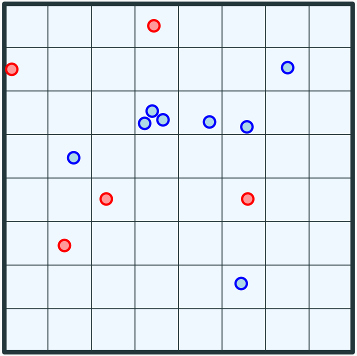
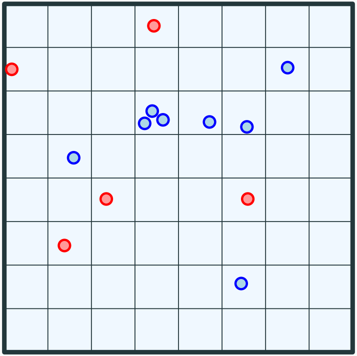
- ...

Warmup: Constant factor approximation for bichromatic closest pair

Bichromatic closest pair



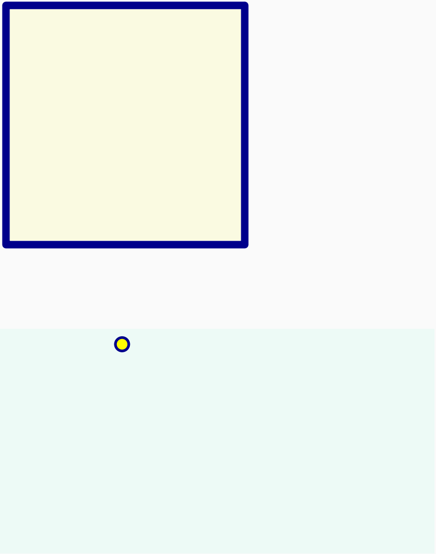
Bichromatic closest pair



Problem (c-approximation)

Maintain a pair (r', b') s.t. $\|r' - b'\| \leqslant c \cdot \min_{(r,b)} \|r - b\|$.

Quadtrees



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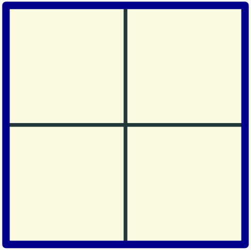

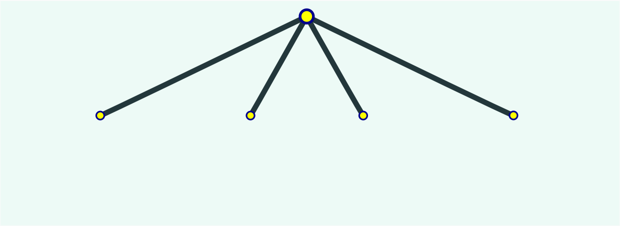

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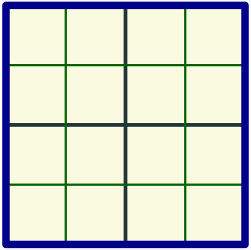

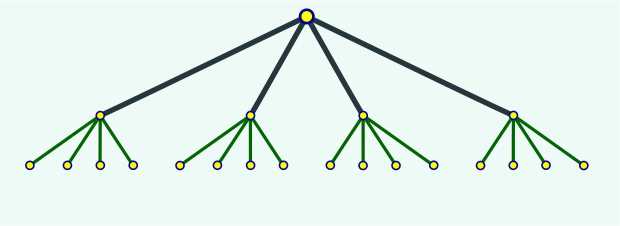

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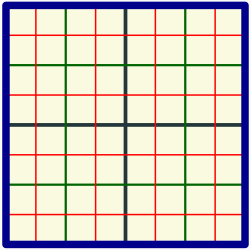

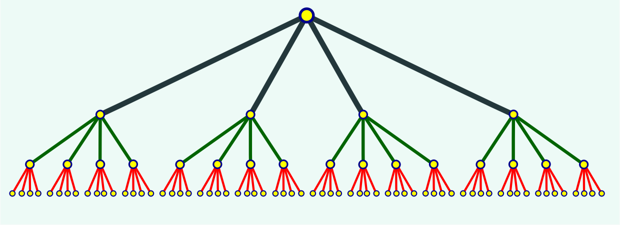

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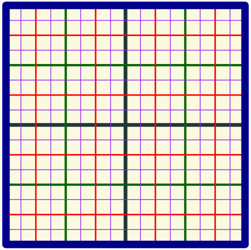
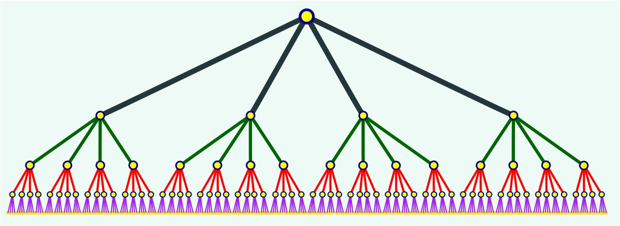
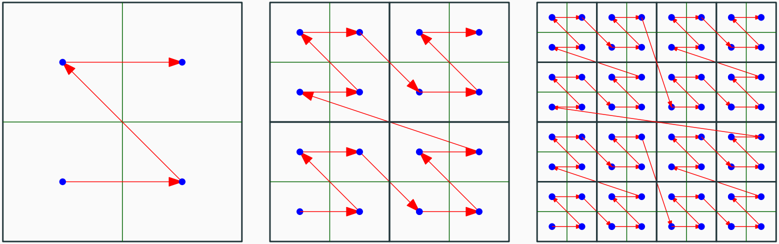




Quadtrees: $\mathbb{Z}$-order

DFS of a quadtree $\implies$ ordering of points ($\mathbb{Z}$-order)



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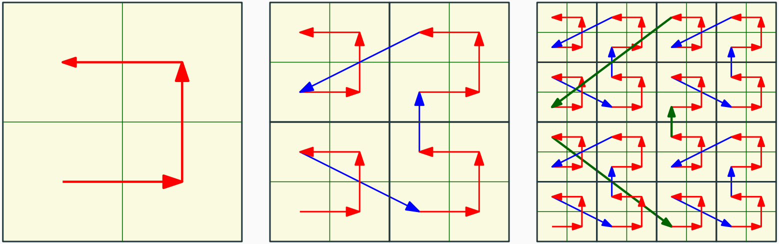
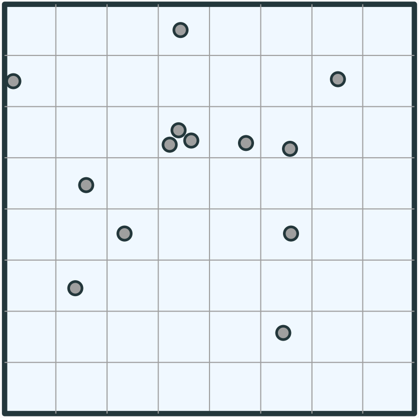


Hope: points close together $\approx$ nearby in ordering



Ordering of points

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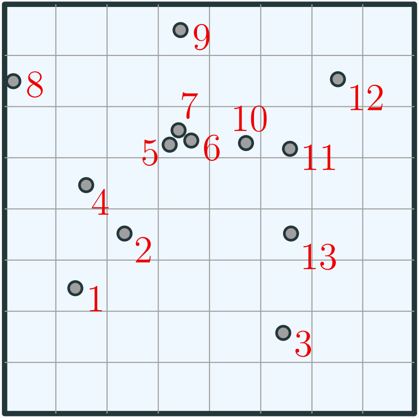
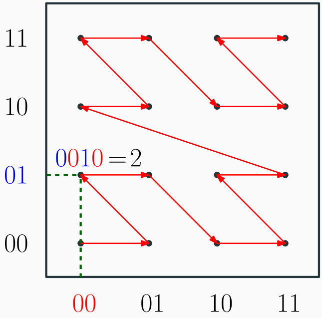


Computing the $\mathbb{Z}$-order

- Let $p = (x, y) \in [2^w] \times [2^w]$
- $x = x_w x_{w-1} \dots x_1$
- $y = y_w y_{w-1} \dots y_1$



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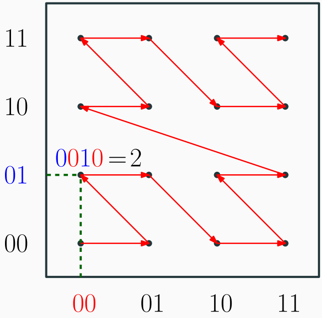
- $\text{shuffle}(p) = y_w x_w y_{w-1} x_{w-1} \dots y_1 x_1$
- Position of p in $\mathcal{Z}$-order = $\text{shuffle}(p)$



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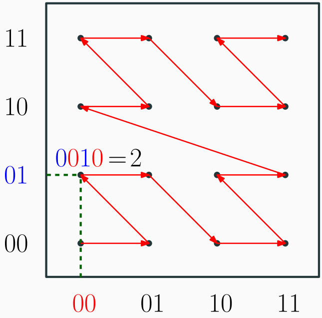
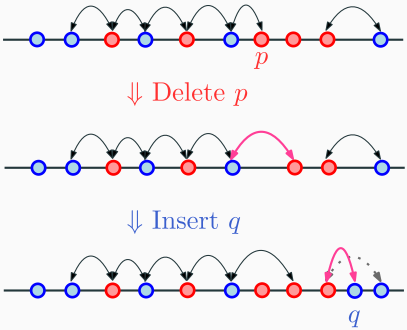


Lemma

$\text{shuffle}(p)$ and $\text{shuffle}(q)$ can be compared with $O(1)$ bitwise-and/xor operations.

Solving the problem in 1D: A solution?

- Map the point set to 1D



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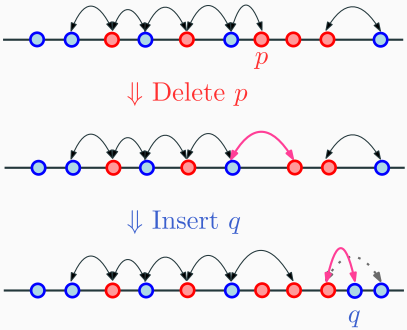

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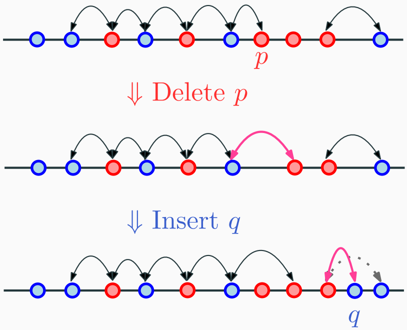

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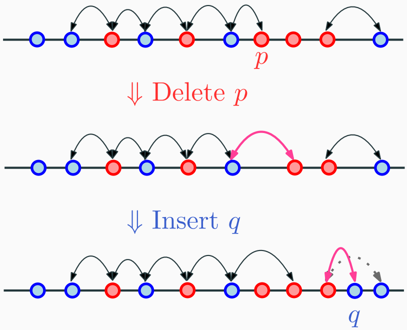

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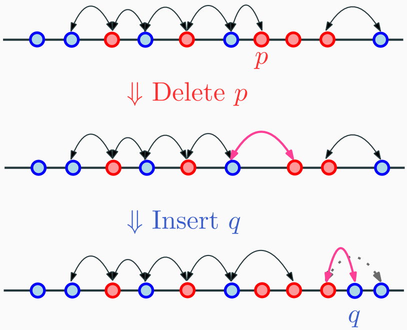
$\Longrightarrow$ Update time $O(\log n)$



Not quite a solution

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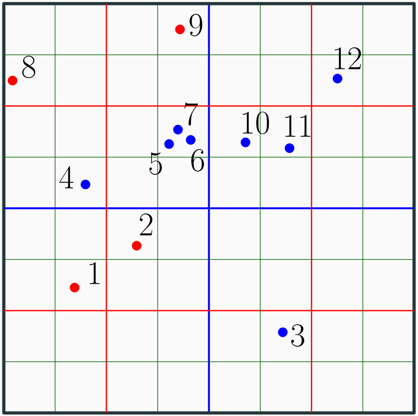

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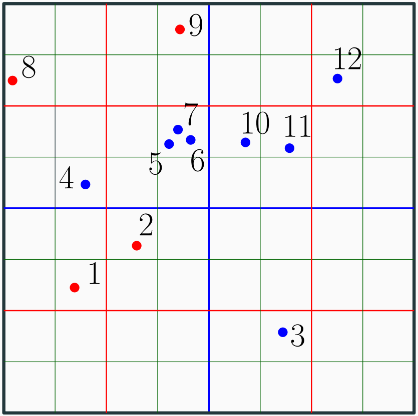
- Idea: Shift the point set



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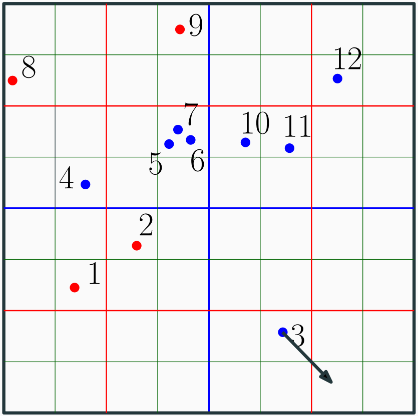

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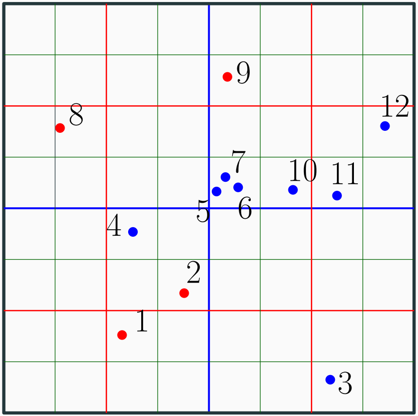

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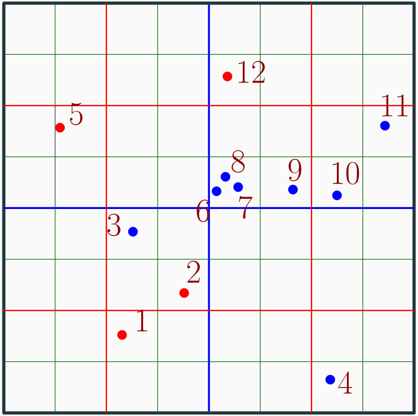

Shifting

Lemma [Chan, 1998]

For $i = 0, \ldots, d$, $v_i = (i/(d+1), \ldots, i/(d+1))$.

For any $p, q \in [0,1)^d$, exists $i \in \{0, \ldots, d\}$ and quadtree cell $\square$:

1. $p + v_i, q + v_i \in \square$
2. $(d+1)\|p - q\| < \text{sidelength}(\square) \leqslant 2(d+1)\|p - q\|$.

A correct solution

- Shift point set $d + 1$ times: $P_0, \ldots, P_d$

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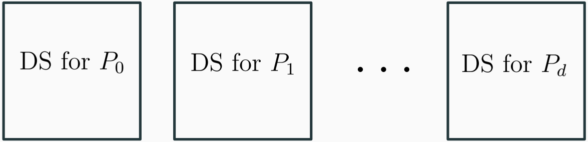

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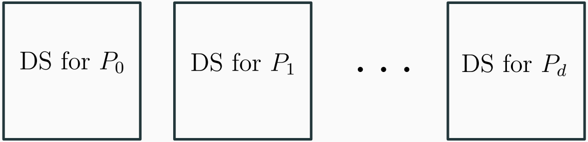

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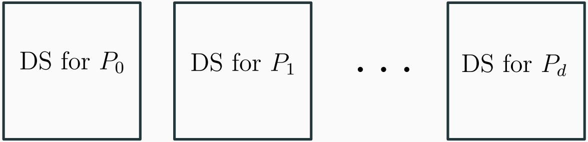

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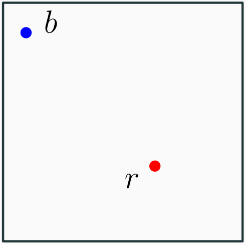
- Claim: $O_d(1)$ approximation

Correctness (cont.)



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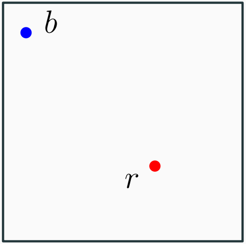
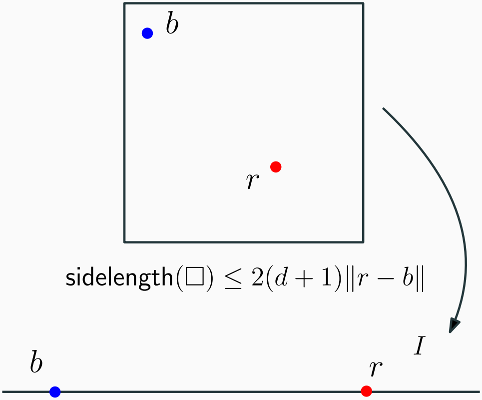


sidelength$(\square) \leq 2(d+1)\|r-b\|$

Correctness (cont.)



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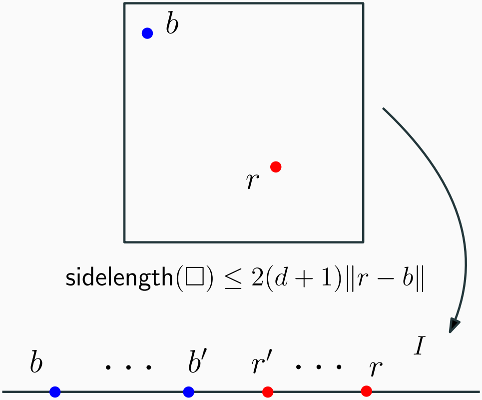

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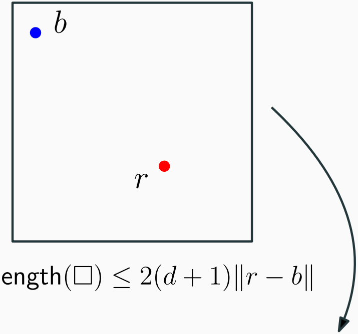

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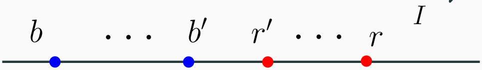

$\|r'-b'\| \leq$ diam$(\square) \leq \sqrt{d} \cdot$ sidelength$(\square) = O_d(1)\|r-b\|$

The challenge: $(1+\varepsilon)$-approximate bichromatic closest pair

Key idea I: Reducing the approximation factor

- Assume $\varepsilon = 2^{-E}$ for $E \in \mathbb{N}$

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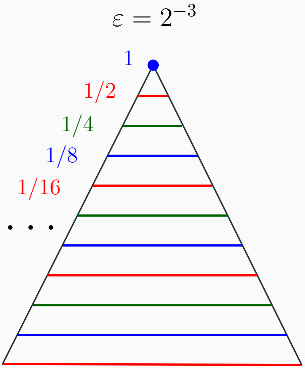

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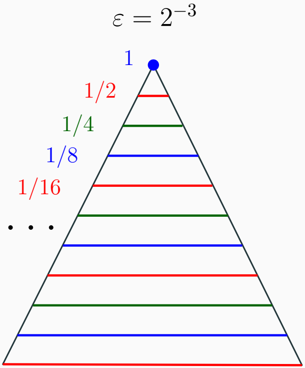

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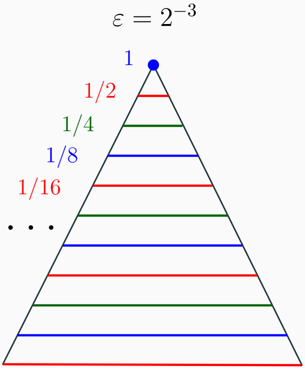
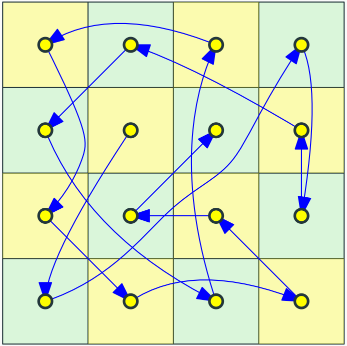
- Can partition a regular quadtree into $\lg(1/\varepsilon)$ ε-quadtrees
- Call them $\mathcal{T}_\varepsilon^1, \ldots, \mathcal{T}_\varepsilon^E$



$O(1)$ problems

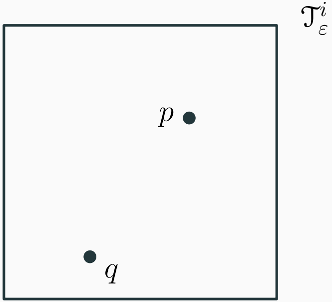
Extend $\mathcal{Z}$-order to ε-quadtrees by ordering $1/\varepsilon^d$ child cells

10	6	9	3
7	1	16	5
11	15	14	4
2	12	8	13



Which order to pick?

$O(1)$ **problems**



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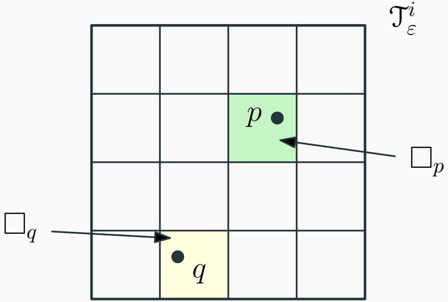

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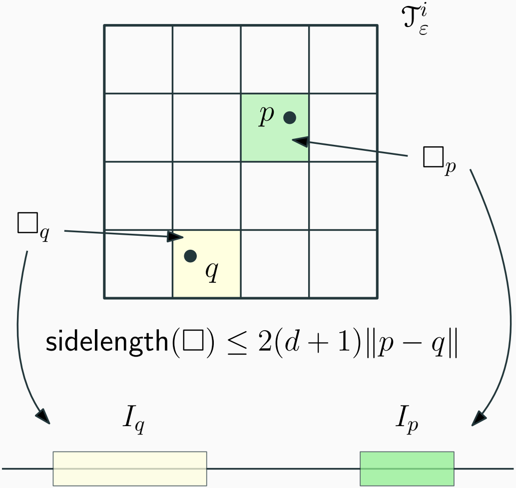


$\mathcal{T}_\varepsilon^i$
p
q
$\square_p$
$\square_q$
sidelength$(\square) \leq 2(d+1)\|p-q\|$
I_q
I_p

Key idea II: Ordering quadtree cells

Problem

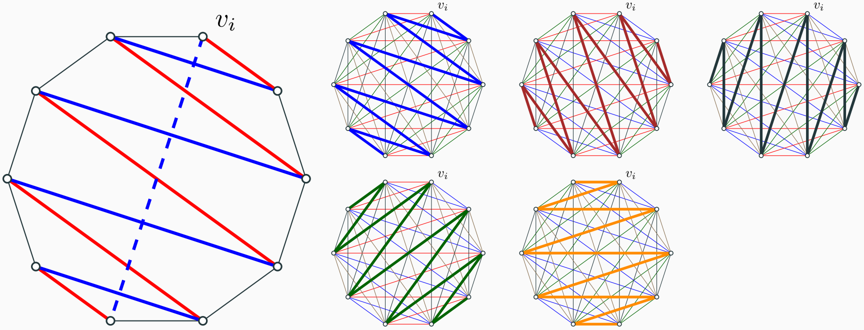
Find a family $\mathfrak{O}$ of orderings of the $1/\varepsilon^d$ cells s.t.:

For any $\square_1, \square_2$, there is an ordering $\sigma \in \mathfrak{O}$ with $\square_1$ adjacent to $\square_2$.

A necessary subproblem

Lemma [Alspach, 2008]

For $[\![n]\!] = \{1, \ldots, n\}$, there are $\lceil n/2 \rceil$ orderings $\mathfrak{O}$ of $[\![n]\!]$ such that for all $i, j \in [\![n]\!]$, $\exists \sigma \in \mathfrak{O}$ where i and j are adjacent in σ.



Ordering quadtree cells

Corollary

There is a set $\mathfrak{O}(\varepsilon)$ of $O(1/\varepsilon^d)$ orderings such that for any $\square_1, \square_2$, there is an order $\sigma \in \mathfrak{O}(\varepsilon)$ where $\square_1$ and $\square_2$ are adjacent.

What we have so far

- $d + 1$ shifted point sets $\equiv d + 1$ quadtrees

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  $\implies O_d\big((1/\varepsilon^d)\log(1/\varepsilon)\big)$ different orderings of P
- Π is this family of locality-sensitive orderings
- For $\sigma \in \Pi$, can decide $p \prec_\sigma q$ with $O(\log(1/\varepsilon))$ bitwise-logical operations.

The solution

- Maintain the 1D data structure for all orderings Π

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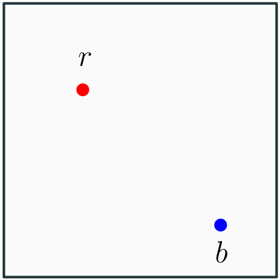
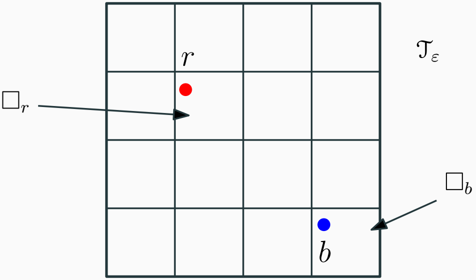
- Space: $O(|\Pi| \cdot n) = O_d((n/\varepsilon^d)\log(1/\varepsilon))$
- Claim: Maintains r', b' with $\|r' - b'\| \leqslant (1+\varepsilon)\|r - b\|$

Correctness



sidelength($\square$) $\leq 2(d+1)\|r-b\|$

Correctness



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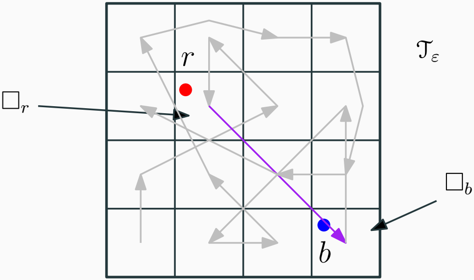

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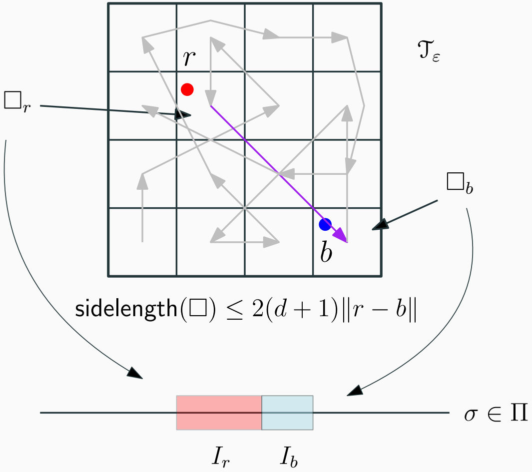
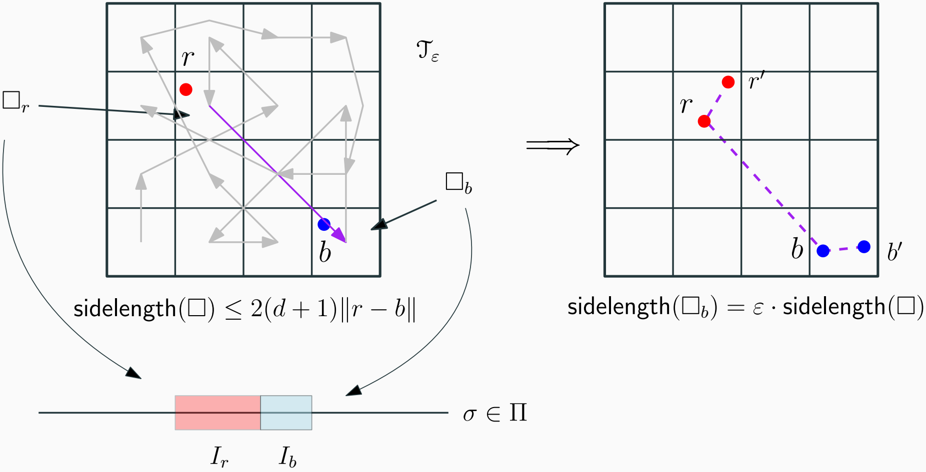

r
b
$\mathcal{T}_\varepsilon$
$\square_r$
$\square_b$
sidelength$(\square) \leq 2(d+1)\|r-b\|$
$\sigma \in \Pi$
I_r
I_b

Correctness



The result

Our result

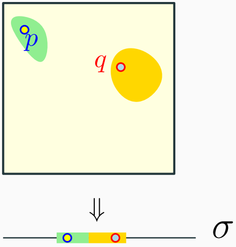
Can maintain the $(1+\varepsilon)$-approximate bichromatic closest pair dynamically with:

1. $O(\log n \log^2(1/\varepsilon)/\varepsilon^d)$ update time
2. $O(n \log(1/\varepsilon)/\varepsilon^d)$ space

Main Theorem

For $\varepsilon \in (0,1)$, there is a set Π of size $O((1/\varepsilon^d)\log(1/\varepsilon))$ s.t. $\forall p, q \in [0,1)^d$, $\exists \sigma \in \Pi$ with:

Points between p and q in σ are distance at most $\varepsilon\|p-q\|$ from p or q.



A simple data structure for dynamic $(1+\varepsilon)$-spanners

Definition

For a set n of P points in $\mathbb{R}^d$ and $t \geqslant 1$, a t-spanner of P is a graph $G = (P, E)$ such that for all $p, q \in P$,

$$\|p - q\| \leqslant \mathrm{dist}_G(p, q) \leqslant t\|p - q\|.$$

Problem

Maintain a $(1 + \varepsilon)$-spanner of P dynamically.

Previous work & result

reference	insertion time	deletion time
[Roditty, 2012]	$O(\log n)$	$O(n^{1/3} \log^{O(1)} n)$
[Gottlieb and Roditty, 2008a]	$O(\log^2 n)$	$O(\log^3 n)$
[Gottlieb and Roditty, 2008b]	$O(\log n)$	$O(\log n)$

Our result

Can dynamically maintain a $(1 + \varepsilon)$-spanner of P with:

1. $O(n \log(1/\varepsilon)/\varepsilon^d)$ edges
2. $O(\log(1/\varepsilon)/\varepsilon^d)$ maximum degree
3. $O(\log n \log^2(1/\varepsilon)/\varepsilon^d)$ update time

Construction

- For each $\sigma \in \Pi$, add edges between consecutive points

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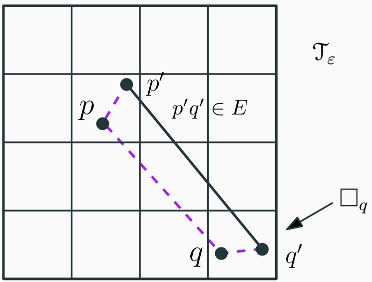
- Update time $O_d((1/\varepsilon^d)\log(n)\log^2(1/\varepsilon))$
- Claim: G is a $(1+\varepsilon)$-spanner

Proof idea

- Proof by induction on length of pairs:
  $\mathrm{dist}_G(p, q) \leqslant (1 + \varepsilon)\|p - q\|$



sidelength($\square$) $\leq 2(d+1)\|p-q\|$

sidelength($\square_q$) $= \varepsilon \cdot$ sidelength($\square$)

Proof idea

- Proof by induction on length of pairs: $\text{dist}_G(p, q) \leqslant (1 + \varepsilon)\|p - q\|$
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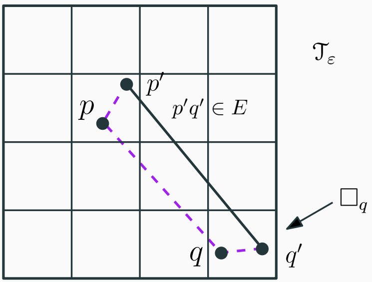

$$\text{sidelength}(\square) \leq 2(d+1)\|p-q\|$$
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- Proof by induction on length of pairs: $\text{dist}_G(p, q) \leqslant (1 + \varepsilon)\|p - q\|$
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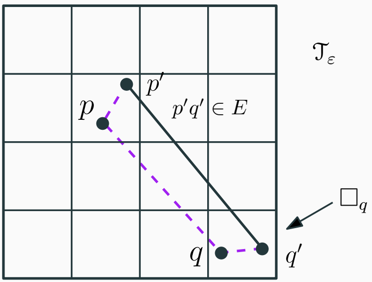
- Readjust ε by c_d



$\text{sidelength}(\square) \leq 2(d+1)\|p - q\|$

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Static & dynamic vertex-fault-tolerant spanners

Fault-tolerant spanners

Definition

For a set of n points P in $\mathbb{R}^d$ and $t \geqslant 1$, a k-vertex-fault-tolerant t-spanner of P is a graph $G = (P, E)$ such that

1. G is a t-spanner, and
2. For any $P' \subseteq P$, $|P'| \leqslant k$, $G \setminus P'$ is a t-spanner for $P \setminus P'$.

Problem

For a static point set P, efficiently construct a “small” k-VFT $(1 + \varepsilon)$-spanner.

Previous work & result

reference	# edges	degree	running time
[Levcopoulos et al., 1998]	$2^{O(k)}n$	$2^{O(k)}$	$O(n\log n + 2^{O(k)}n)$
	$O(k^2 n)$	unbounded	$O(n\log n + k^2 n)$
	$O(kn\log n)$	unbounded	$O(kn\log n)$
[Lukovszki, 1999]	$O(kn)$	$O(k^2)$	$O(n\log^{d-1} n + kn\log\log n)$
[Czumaj and Zhao, 2004]	$O(kn)$	$O(k)$	$O(kn\log^d n + k^2 n\log k)$
[Chan et al., 2015]	$O(k^2 n)$	$O(k^2)$	$O(n\log n + k^2 n)$
[Kapoor and Li, 2013] & [Solomon, 2014]	$O(kn)$	$O(k)$	$O(n\log n + kn)$

Our result

A k-VFT $(1+\varepsilon)$-spanner of P with

1. $O(kn\log(1/\varepsilon)/\varepsilon^d)$ edges
2. $O(k\log(1/\varepsilon)/\varepsilon^d)$ maximum degree
3. $O\big((n\log n\log(1/\varepsilon) + kn)\log(1/\varepsilon)/\varepsilon^d\big)$ construction time

Construction

- For each $\sigma \in \Pi$ and each $p \in P$, connect p to its $k+1$ predecessors and successors in σ

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Construction

- For each $\sigma \in \Pi$ and each $p \in P$, connect p to its $k + 1$ predecessors and successors in σ
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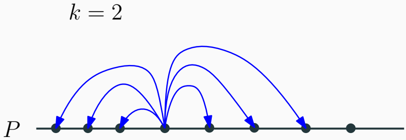
Sketch proof

- G is a $(1+\varepsilon)$-spanner



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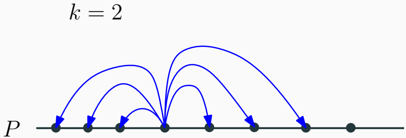
- G is a $(1+\varepsilon)$-spanner
- Consider $P' \subseteq P$, $|P'| \leqslant k$



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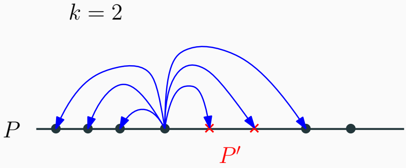

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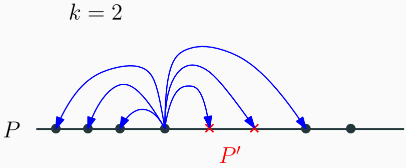

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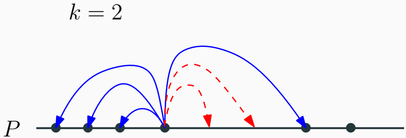

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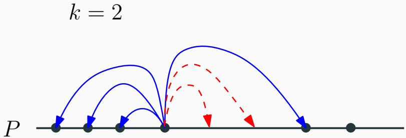

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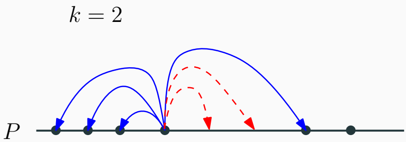
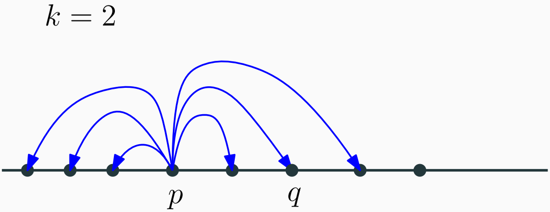


Update time

Any update changes $O(k)$ edges in G



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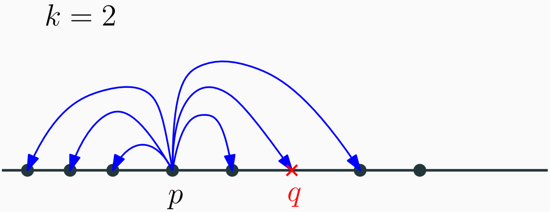

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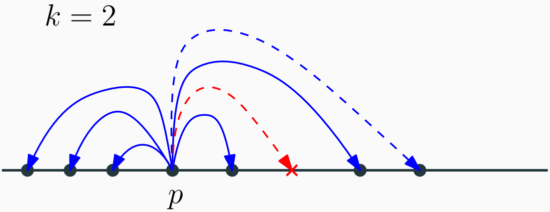

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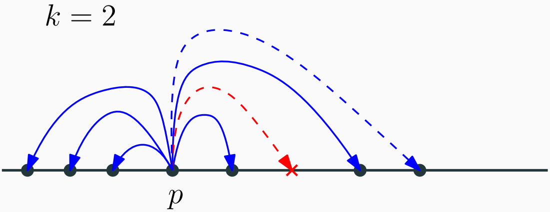

Update time $O_d((\log n \log(1/\varepsilon) + k)|\Pi|)$

Result

Our result

A k-VFT $(1+\varepsilon)$-spanner of P with

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2. $O(k \log(1/\varepsilon)/\varepsilon^d)$ maximum degree
3. $O\big((n \log n \log(1/\varepsilon) + kn) \log(1/\varepsilon)/\varepsilon^d\big)$ construction time

New: Can also maintain dynamically with update time

$$O\Big((\log n \log(1/\varepsilon) + k) \log(1/\varepsilon)/\varepsilon^d\Big).$$

Conclusion

Main Theorem

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Remarks

- Extends to $\| \cdot \|_p$ norms

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Remarks

- Extends to $\|\cdot\|_p$ norms
- “Replacement” for well-separated pair decomposition
- $\approx$ locality-sensitive hashing (smaller family of orders, weaker guarantees)

Applications

1. Approximate bichromatic closest pair: Improved update time $\approx O(\log^3 n)$ [Eppstein, 1995] $\rightarrow O(\log n)$

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7. Static robust $(1+\varepsilon)$-spanners: See [Buchin et al., 2018]

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