Active Learning a Convex Body in Low Dimensions

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UIUC Theory Seminar, November 11, 2019

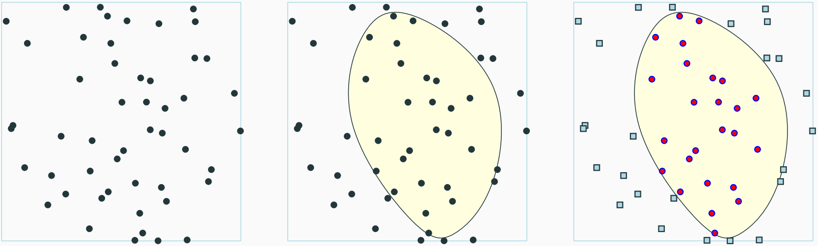
An innocent problem

Problem

Input: $P \subset \mathbb{R}^2$, oracle for unknown convex body C.

Oracle: Query $q \in \mathbb{R}^2$, returns true $\iff q \in C$.

Goal: Compute $P \cap C$ using fewest number of oracle queries.

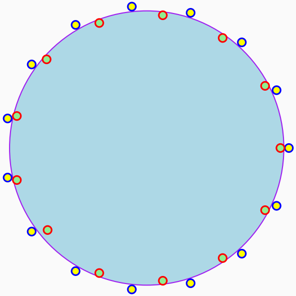


Motivation: Active learning

- Input space X
- Learner data: $x_1, \ldots, x_n \in X$ (without labels)
- Learner can query oracle for label of any $q \in X$
- Build classifier using few queries
- What queries to choose?

Bad news

- Worst case: query all points
- **Question:** More interesting model to study?

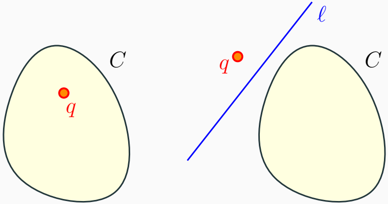


Modified problem

Problem

Input: $P \subset \mathbb{R}^2$, oracle for unknown convex body C.

Oracle: Separation oracle



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  - Proximity probe [Panahi, Adler, et al., 2013]
- Minimizing communication complexity

One approach: PAC learning

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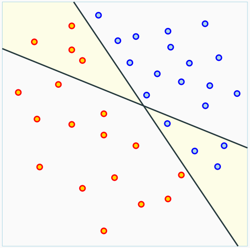
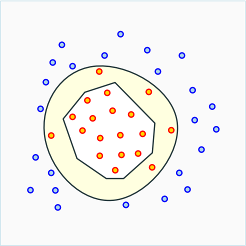
- Size of sample?

One approach: PAC learning

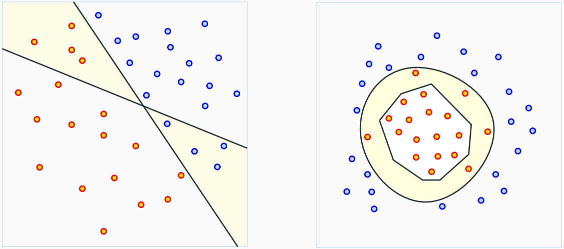
- Misclassified points = symmetric difference of learned and true classifier





One approach: PAC learning

- Misclassified points = symmetric difference of learned and true classifier
- Halfplane $\implies$ symmetric difference is a wedge



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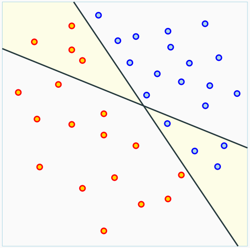

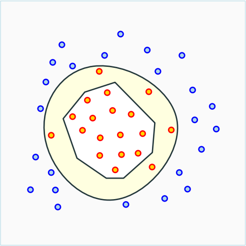

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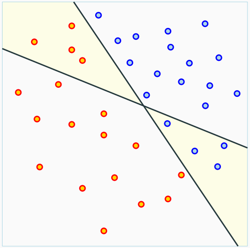
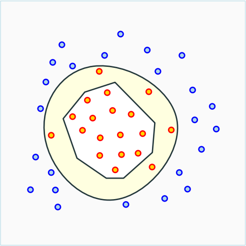
- Scheme fails for arbitrary convex regions





Hard vs. easy instances

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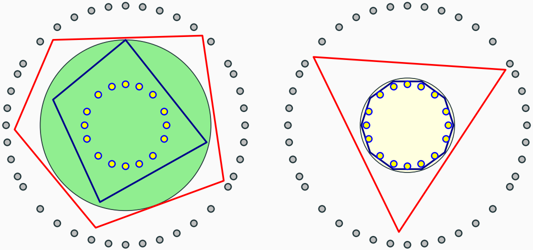
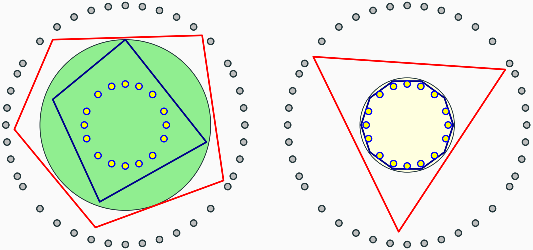
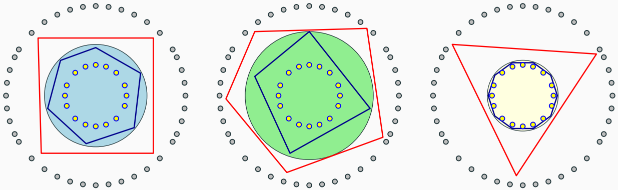


Hard vs. easy instances

- Worst case: query all points
- **Goal:** design instance sensitive algorithms



A lower bound



- F_{in} = convex polygon with fewest vertices s.t. $F_{\text{in}} \subseteq C$ and $C \cap P = F_{\text{in}} \cap P$.

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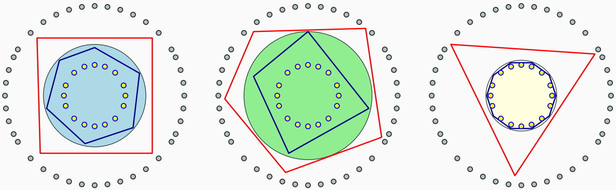

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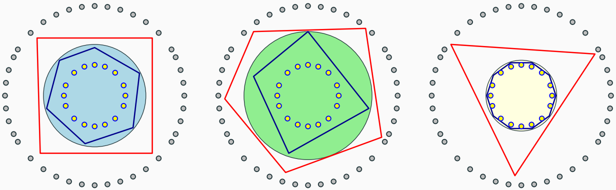

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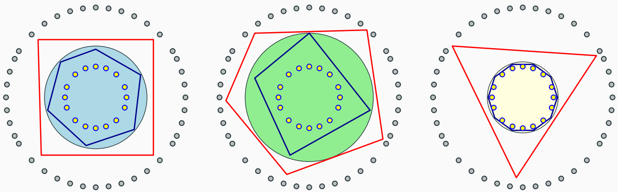

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Lemma

Any algorithm must make at least $\sigma(P, C)$ oracle queries.

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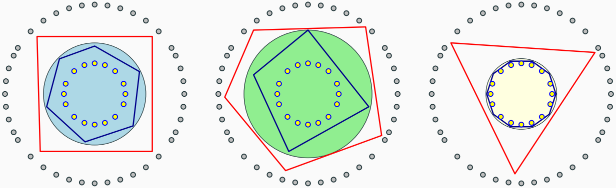

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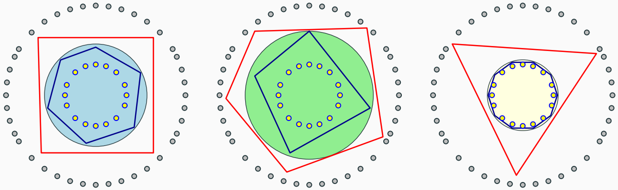

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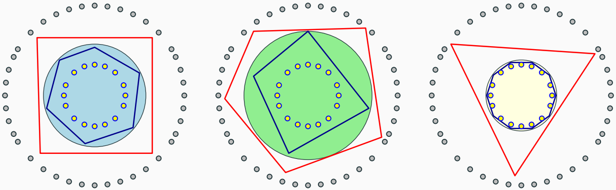

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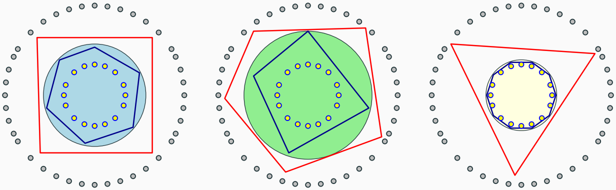

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  $\implies |Q_{\text{in}}| \geqslant |K| \geqslant |F_{\text{in}}|$ □

Results

Problem	Lowerbound	Upperbound
Classify (2D)	$\sigma(P, C)$	$O(k(P) \log n)$ (†)

(†) $k(P) =$ largest # of pts of P in convex position

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First attempt: A greedy algorithm

The greedy algorithm: preliminaries

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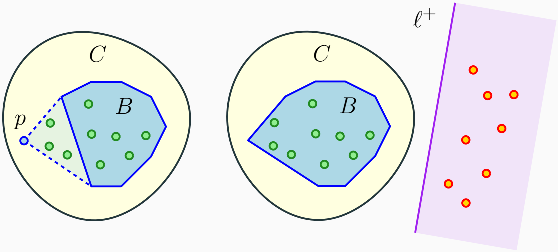

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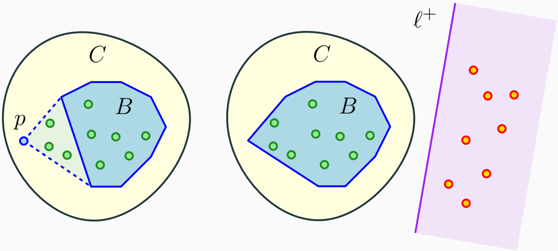
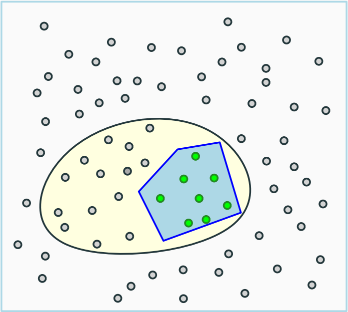
- $c \in \mathbb{R}^2$ is a centerpoint for P if for all halfspaces ℓ^+:
  $c \in \ell^+ \implies |P \cap \ell^+| \geqslant |P|/3$.



The greedy algorithm

$U \subseteq P$ unclassified points. While $U \neq \varnothing$:



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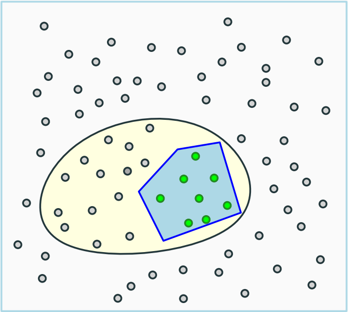

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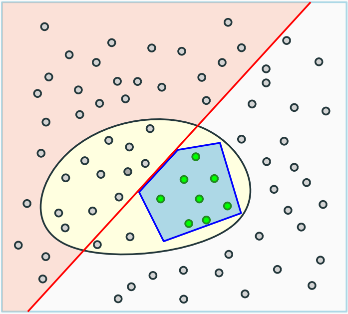

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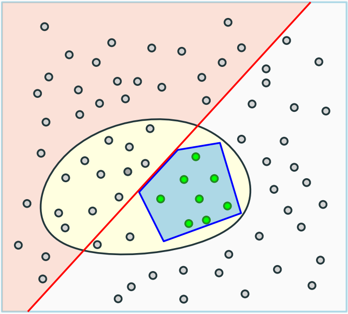

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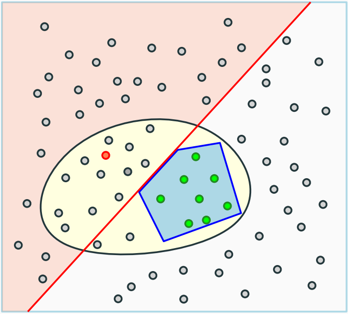

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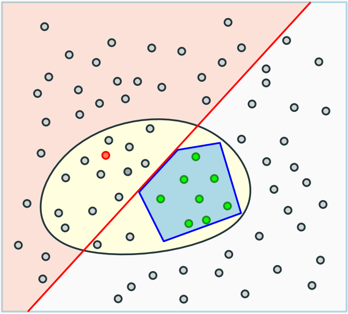

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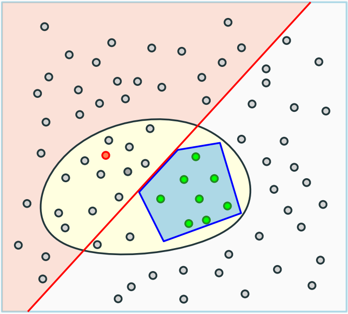

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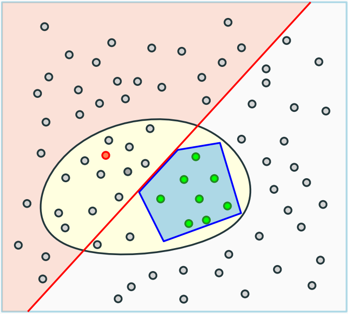

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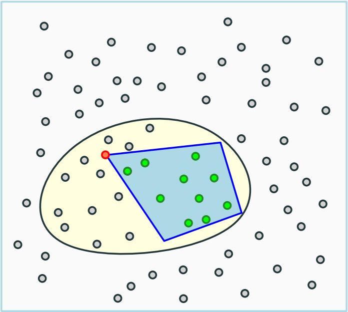
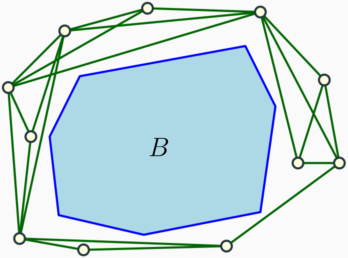


Animation

Analysis

- Count visible pairs of points



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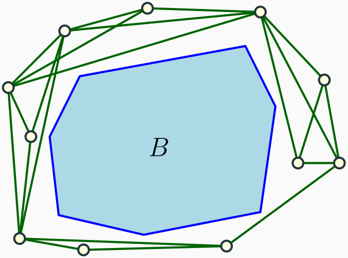
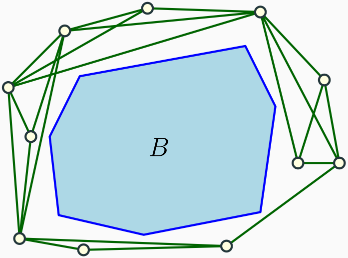
- In each iteration:



Analysis

- Count visible pairs of points
- In each iteration:
  - (A) Pairs lose visibility



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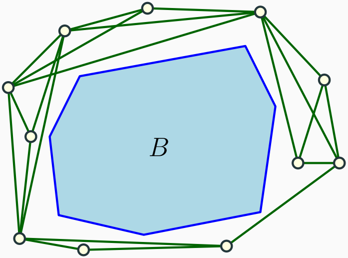

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- In each iteration:
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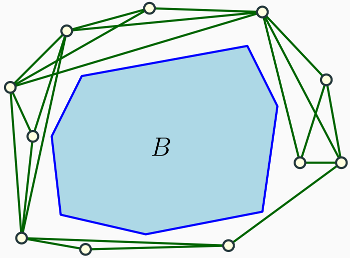
Lemma

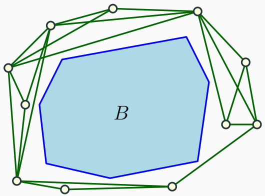
Number of visible pairs decrease by a (roughly) constant fraction in each iteration.



Two interpretations of the visibility graph

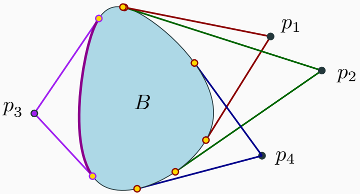
Visibility graph $G_B = (P, E)$:

$(p, q) \in E \iff pq \cap B = \varnothing$



$p \in P$ has interval $I(p)$

$(p, q) \in E \iff I(p) \cap I(q) \neq \varnothing$



Two observations

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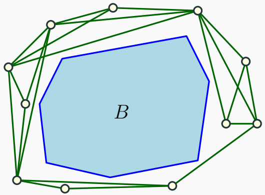
1. $Q \subseteq P$ independent set in $G_B \implies Q$ is in convex position

$$(p, q) \in E \iff pq \cap B = \emptyset$$



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1. $Q \subseteq P$ independent set in $G_B \implies Q$ is in convex position
2. $Q \subseteq P$ and B are linearly separable $\implies Q$ clique in G_B

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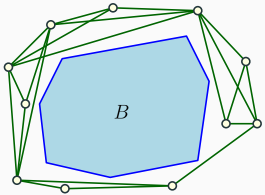

Lemma 1

$\alpha(G_B)$ = size of largest indep. set, $\omega(G_B)$ = maximum depth, then $|E| = O(\alpha(G_B)\omega(G_B)^2)$.

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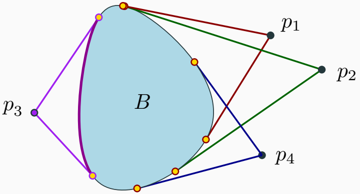

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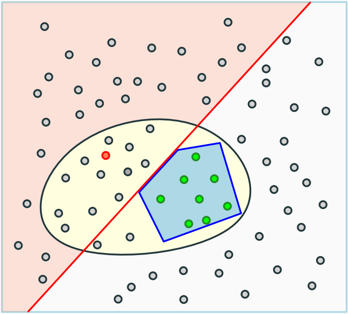

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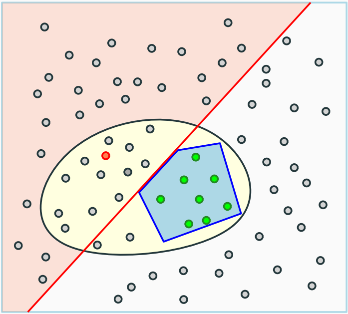

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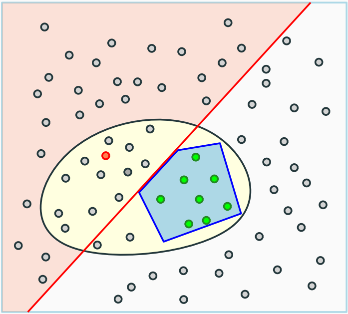

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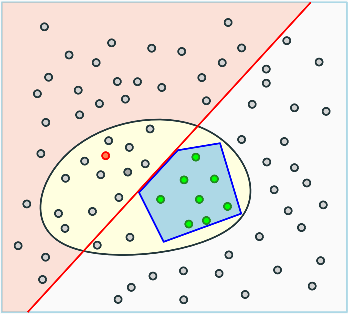
When $c \in C$, **expand**(c) deletes $\geqslant \omega(G_B)^2/36$ edges from G_B.



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Extending the algorithm to 3D

Extending the algorithm

$U \subseteq P$ unclassified points. While $U \neq \varnothing$:

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Everything still works

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An instance optimal algorithm in 2D

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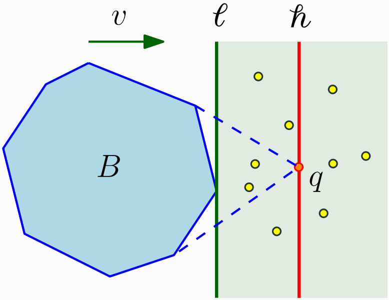
Directional climbs

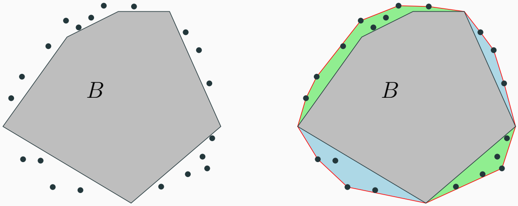
Given direction v:

- Compute line ℓ tangent to B, perpendicular to v
- Regular iteration on $\ell^+ \cap U$.



Pockets

Pocket: A connected region of $\mathrm{CH}(U \cup B) \setminus B$



Lemma

In $O(\log n)$ oracle queries, can split a pocket Υ, into two pockets Υ_1, Υ_2, $|\Upsilon_i \cap P| \leqslant (2/3)|\Upsilon \cap P|$.

Algorithm sketch

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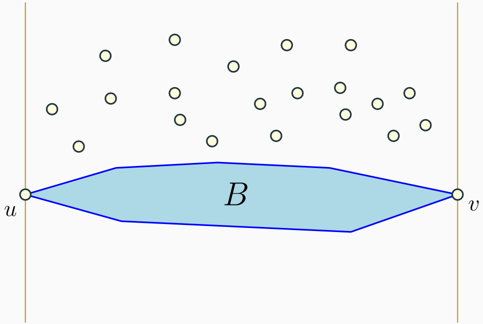

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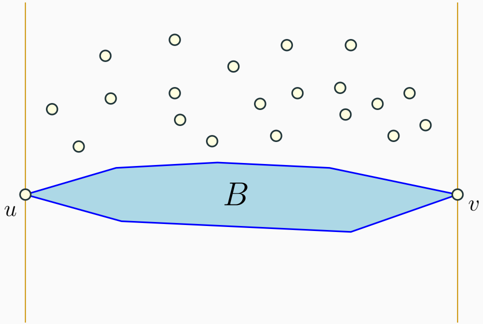

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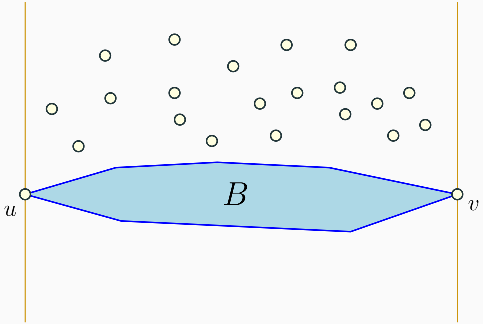

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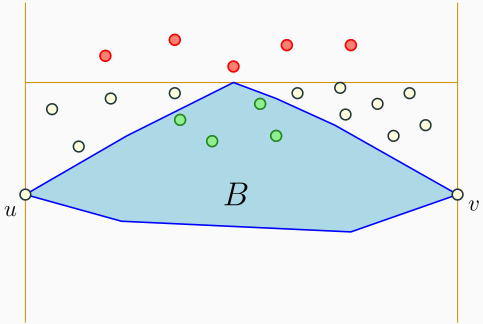

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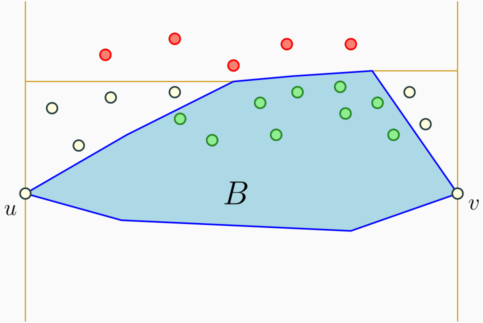

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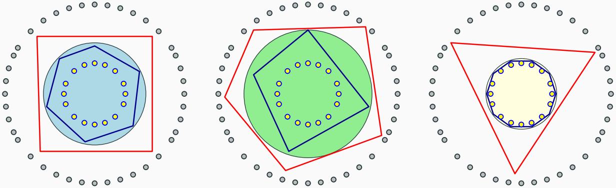

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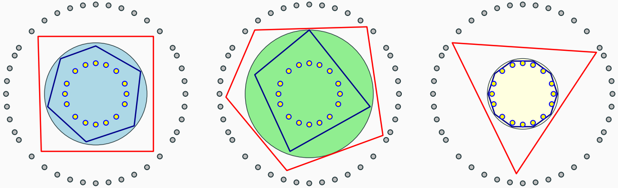

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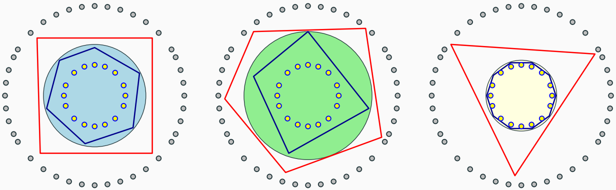

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Can classify all points using $O(\sigma(P, C) \log^2 n)$ oracle queries.

Conclusions

Conclusion & open problems

Problem	Lowerbound	Upperbound
Classify (2D)	$\sigma(P, C)$	$O(k(P) \log n)$ $O(\sigma(P, C) \log^2 n)$
Classify (3D)	—	$O(k(P) \log n)$
Verify in	$\lvert F_{\mathrm{in}} \rvert$	$O(\lvert F_{\mathrm{in}} \rvert \log n)$
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- Conjecture: Greedy extends to $\mathbb{R}^d$ ($d \geqslant 3$), queries depend exponentially on d

References i

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